

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
Department of Mechanical Engineering  
**2.004 Dynamics and Control II**  
Fall 2007

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Problem Set #5

**Solution**

Posted: Friday, Oct. 19, '07

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1. Modify the Laplace-domain KVL equation (2) to include the inductance. Argue that (4), (6), (9–10) need not be modified in this system, and thus (11) remains valid.

*Answer:* When we include the inductance  $L$  in our model, we have to take into account the voltage drop across the inductor in the KVL equation.

$$V_s(s) = I(s)R + I(s)Ls + K_v\Omega(s).$$

In the mechanical part, the torque generated by the motor drives the same inertias and has to overcome the same friction; that is, the mechanical part doesn't change. Therefore (4), (6), (9-10) need not be modified and (11) remains valid.

2. Substitute (11) to your new KVL equation from question 1 and derive the open-loop transfer function  $V(s)/V_s(s)$ .

*Answer:* Inserting (11) to the new KVL relation in question 1, we obtain

$$\begin{aligned} V_s(s) &= \left( \frac{J + r^2M}{rK_m} s + \frac{D + r^2f_v}{rK_m} \right) (R + Ls) V(s) + K_v \frac{V(s)}{r} \\ &= \left\{ \frac{J + r^2M}{rK_m} Ls^2 + \left( \frac{(J + r^2M)R + (D + r^2f_v)L}{rK_m} \right) s + \frac{D + r^2f_v}{rK_m} R + \frac{K_v}{r} \right\} V(s). \end{aligned}$$

The transfer function is

$$\frac{V(s)}{V_s(s)} = \frac{\frac{rK_m}{L(J + r^2M)}}{s^2 + \left( \frac{R}{L} + \frac{D + r^2f_v}{J + r^2M} \right) s + \frac{(D + r^2f_v)R + K_mK_v}{L(J + r^2M)}}.$$

3. Substitute  $L = 1$  H and the remaining numerical values from the Supplement, and find the poles of your transfer function. You should find that this system is *underdamped*.

*Answer:* Using numerical values given in the supplement, the transfer function is simplified to

$$\frac{V(s)}{V_s(s)} = \frac{0.3162/L}{s^2 + (1/L + 1)s + 2/L}.$$

Here  $L = 1$  H, so the transfer function is

$$\frac{V(s)}{V_s(s)} = \frac{0.3162}{s^2 + 2s + 2}$$

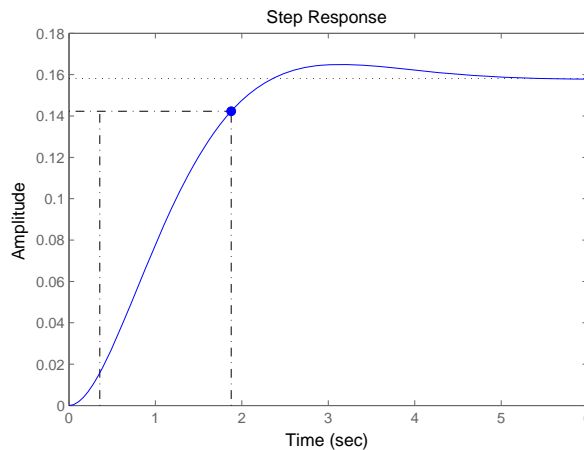
Poles:  $p_1 = -1 + j$ ,  $p_2 = -1 - j$ .

Natural frequency  $\omega_n = \sqrt{2}$  rad/s.

Damping ratio  $\zeta = 2/(2\omega_n) = 2/(2\sqrt{2}) = 1/\sqrt{2} = 0.7071 < 1$ : underdamped.

4. Use the LTI tool to plot the step response of the open-loop system. Compute the steady-state analytically using the transfer function from question 3 and verify that it matches the LTI tool's prediction.

*Answer:*



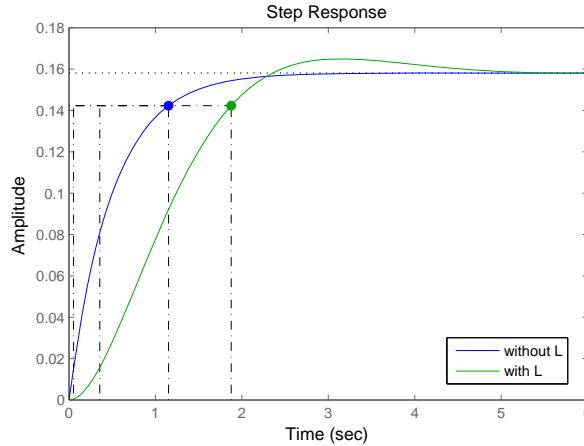
From the final value theorem,

$$v_\infty = \lim_{s \rightarrow 0} s \left( \frac{1}{s} \right) \frac{0.3162}{s^2 + 2s + 2} = \frac{0.3162}{2} = 0.1581.$$

5. Use the plot from question 4 to estimate the rise time of the system. How does it compare with the rise time of the open-loop system from the Supplement (*i.e.*, the identical system but without the inductance.)

*Answer:* From the step response plot,  $T_r \approx 1.5$  (s). (The normalized rise time method in Nise textbook gives you the same result.) The rising time of the inductance-free system is 1.1 sec from the following step response; (It also can be computed as  $T_r = 2.2\tau$  from the lecture note 7 page 1)

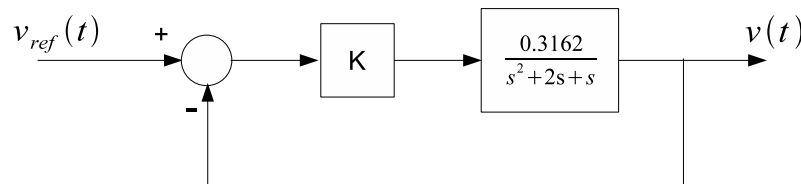
The time constant of the inductance-free system was 0.5 sec. But that's not a fair comparison, since the time constant of the 1<sup>st</sup>-order system is the time from 0 to 63.2% of the steady-state value, whereas the rise time of a 2<sup>nd</sup>-order system is the time from 10% to 90% of steady-state. However, we can use the LTI tool to easily verify that the inductance-free system was faster; see the plot below where the two responses (with and without  $L$ ) have been superimposed.



The physical explanation for the slowness of the 2<sup>nd</sup>-order system is clear once we realize that the inductor is an energy storage element. It “wants” to store energy in its magnetic field, and thus slows down the application of current to the motor which ultimately results in acceleration. If we look more carefully in the detailed dynamics of the electrical circuit, the voltage across the inductor is  $Ldi/dt$ . Upon application of a step voltage source to the DC motor circuit, the current rises rapidly, but at the same time the *derivative* of the current results in a large voltage drop across the inductor. However, the current itself is determined by the voltage across the *resistor* (recall  $i = v_R/R$ ). Since the inductor consumes a large fraction of the potential provided by the source, the resistor is left with a smaller fraction, and thus the current through the motor and the acceleration are reduced. This is why inserting the inductor results in a slower system response.

- Using the numerical form of the open-loop transfer function from question 3, derive the closed-loop transfer function with the feedback gain  $K$  as a variable.

*Answer:* The block diagram of the system is shown below:



The closed-loop transfer function is

$$\frac{V(s)}{V_{ref}(s)} = \frac{0.3162K}{s^2 + 2s + (2 + 0.3162K)}$$

- Compute the steady-state velocity and the steady-state error as function of feedback gain  $K$  for the closed-loop system of question 6. Use an argument based on the *physics* of the system to explain similarities or differences to the inductance-free system that was analyzed in the Supplement.

*Answer:* Using the Final Value Theorem, we can compute the steady-state velocity.

$$v_{\infty} = \lim_{s \rightarrow 0} s \frac{1}{s} \frac{0.3162K}{s^2 + 2s + (2 + 0.3162K)} = \frac{0.3162K}{2 + 0.3162K}.$$

Hence the steady-state error is

$$e_{\infty} = 1 - v_{\infty} = \frac{2}{2 + 0.3162K}.$$

The two systems have the same steady-state velocity and steady-state error. At steady-state, the current is constant, and therefore there is no voltage drop across the inductor (recall that the voltage across the inductor is proportional to the derivative of the current.) The current is then determined by the resistor alone, just as in the case of the 1<sup>st</sup>-order system. So basically at steady-state the dynamics of the 2<sup>nd</sup>-order system are identical to those of the 1<sup>st</sup>-order system because the inductor has “dropped out of the picture.” However, the transient response is a little bit different as we saw in question 5.

8. Substitute  $K = 1$  into the closed-loop transfer function from question 6 and use Nise section 4.6 to estimate the peak time, percent overshoot and settling time.

*Answer:* The closed-loop transfer function is

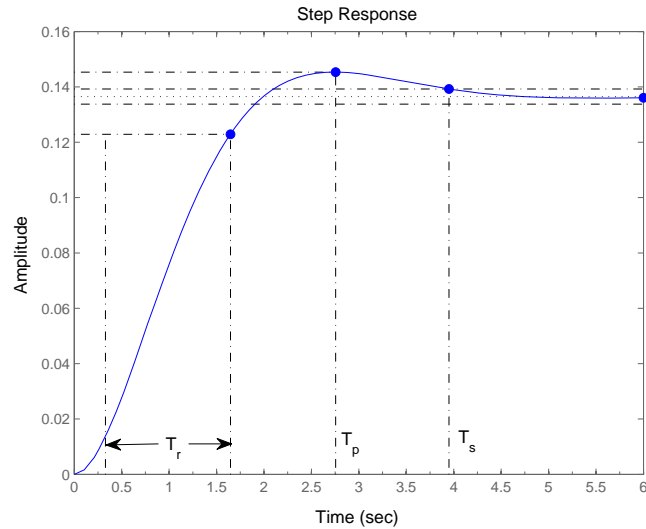
$$\frac{V(s)}{V_{ref}(s)} = \frac{0.3162}{s^2 + 2s + 2.3162},$$

where  $\omega_n = \sqrt{2.3162} = 1.5219$  (rad/s) and  $\zeta = 2/(2\omega_n) = 0.6572$ .

- $T_p = \pi/(\omega_n \sqrt{1 - \zeta^2}) = 2.7383$  (s)
- %OS =  $\exp\left(-\frac{\zeta\pi}{\sqrt{1 - \zeta^2}}\right) \times 100 = 6.4677$  %
- $T_s = 4/(\zeta\omega_n) = 4$  (s)

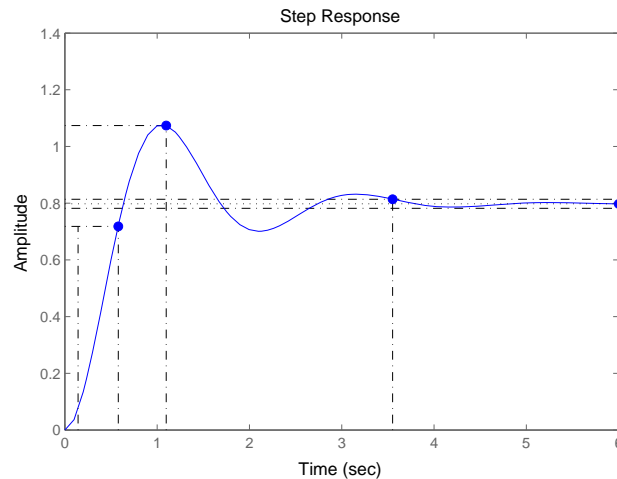
9. Use the LTI tool to plot the step response of the closed loop system with  $K = 1$  and verify that the steady-state velocity, peak time, percent overshoot and settling time that you obtain from the numerical simulation are in agreement with the analytical estimates.

*Answer:*



10. Substitute  $K = 25$  into the closed-loop transfer function from question 6, and use the LTI tool to plot the step response of this high-gain system. Comment, based on the plot and as thoroughly as you can, on the similarities and differences, if any, to the low-gain system step response from questions 8–9.

*Answer:*



As gain becomes bigger, %OS becomes larger.  $T_r$  and  $T_s$  becomes smaller, *i.e.* the system response becomes faster. The steady-state velocity is also getting bigger and steady state-error becomes smaller. Physically, the speed-up happens because larger gain means that the system becomes more responsive to changes in its own output (via the high-gain feedback loop). Another way to come to the same conclusion from the transfer function is that increasing the gain results in increasing the natural frequency of the underdamped system; therefore, overshoot will increase and the transient speed will increase (become faster.)

11. Now go back to your open-loop transfer function from question 2 and substitute

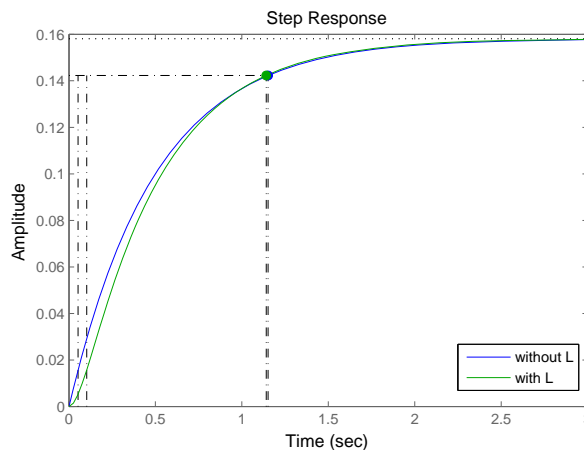
$L = 1/16 = 0.0625$  H. Find the poles of the new transfer function; you should observe that this system is *overdamped*. Comment on the locations of the two poles relative to the pole of the inductance-free system from the Supplement.

*Answer:* With  $L = 1/16$  H, the open-loop transfer function is

$$\frac{V(s)}{V_s(s)} = \frac{5.0592}{s^2 + 17s + 32}.$$

Poles:  $p_1 = -2.16$  and  $p_2 = -7.42$ . The system is overdamped because it has two real negative poles.

The pole of the inductance-free system is  $p = -2$ . The fast pole of the system with inductance is actually slightly *faster* than the one of the inductance-free system. ( $-2.16$  is to the left of  $-2$ ). However, the overdamped system with inductance is actually slightly *slower* than the system without inductance, as we can see in the comparative plot below (which should be compared also with the plot we provided in response to question 5.) Energy storage in the inductor, just as in the case of the underdamped system, makes the overdamped system slower than the inductor-free 1<sup>st</sup>-order system. The reason the response of the overdamped system is only slightly slower than that of the first-order system, whereas the response of the underdamped system was significantly slower than the first-order system is the lack of oscillations in the overdamped system. In the underdamped system, the presence of oscillations keeps the derivative of the current at a large value for a longer time.



12. Using the numerical form of the open-loop transfer function from question 11, derive the closed-loop transfer function with the feedback gain  $K$  as a variable.

*Answer:* The closed-loop transfer function is

$$\frac{V(s)}{V_{ref}(s)} = \frac{5.0592K}{s^2 + 17s + (32 + 5.0592K)}.$$

- 13.** If the feedback gain  $K$  were to vary from 0 to  $+\infty$ , would the system be **(i)** always overdamped; **(ii)** always underdamped; **(iii)** transition to/from overdamped from/to underdamped? Justify your answer using the transfer function you derived in question 12.

*Answer:* **(iii)** transition from overdamped to underdamped.

The poles are

$$p_{1,2} = \frac{-17 \pm \sqrt{17^2 - 4(32 + 5.0592K)}}{2}.$$

Therefore the locations of poles are dependent of  $K$ .

If  $4(32 + 5.0592K) < 17^2$ , two real poles; overdamped.

If  $4(32 + 5.0592K) = 17^2$ , one real pole(double); critically damped.

If  $4(32 + 5.0592K) > 17^2$ , two poles with imaginary part; underdamped.

So the correct answer is (iii) with a transition from overdamped to underdamped when  $K \approx 7.956$ .

- 14.** Compute the steady-state velocity and the steady-state error as function of feedback gain  $K$  for the closed-loop system of question 12. Are they the same or different than those of the high-inductance closed-loop system from question 7? Is that consistent with the physics-based argument that you made in your answer to question 7?

*Answer:* Using the Final Value Theorem, we can compute the steady-state velocity.

$$v_\infty = \lim_{s \rightarrow 0} s \left( \frac{1}{s} \right) \frac{5.0592K}{s^2 + 17s + (32 + 5.0592K)} = \frac{5.0592K}{32 + 5.0592K}.$$

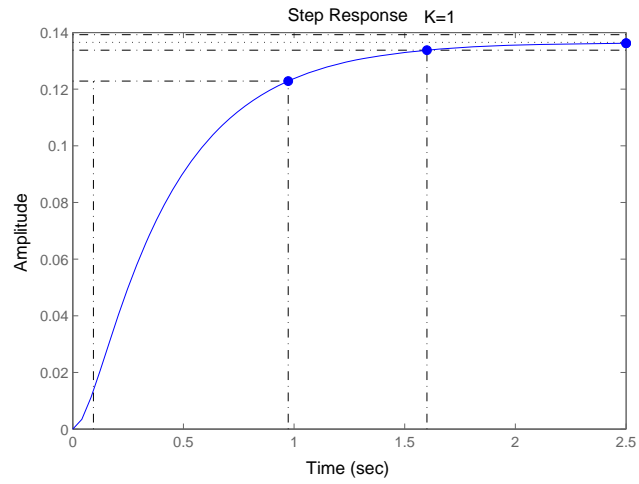
Hence the steady-state error is

$$e_\infty = 1 - v_\infty = \frac{32}{32 + 5.0592K}.$$

We can easily see that these are identical to the answers to question 7, and this is no surprise: just as in the case of the underdamped circuit, the inductor does not participate in the steady-state dynamics because when the current is constant, the voltage drop across the inductor is zero. Therefore at steady-state the 2<sup>nd</sup>-order system dynamics are identical to the 1<sup>st</sup>-order system.

- 15.** Use the LTI tool to plot the step response of the closed-loop system from question 12 for gains  $K = 1$  and  $K = 25$ . Verify that the responses generated by the LTI tool are consistent with your answer to questions 13, 14.

*Answer:* If  $K=1$ ,  $p_1 = -2.5678$  and  $p_2 = -14.4322$ . The system is overdamped. Steady state = 0.1365.



If  $K=25$ ,  $p_{1,2} = -8.5 \pm j9.286$ . The system is underdamped. Steady state = 0.7981.

