2.58 Spring 2006
Midterm 1 Solutions

Question 1.

The peak wavelength of the solar radiation is around 500 nm. For a particle with a diameter d less than 50 nm, Rayleigh scattering is a good approximation. The absorption efficiency of the small particle is given by

$$Q_a = 4x \,\mathrm{Im}\left(\frac{m^2-1}{m^2+2}\right) = \frac{4\pi d}{\lambda}\mathrm{Im}\left(\frac{(2+0.1i)^2-1}{(2+0.1i)^2+2}\right) = \frac{0.133\pi d}{\lambda} \tag{1}$$

The solar irradiance that reaches the particle is

$$G_\lambda = \left(\frac{R_s}{d_{se}}\right)^2 e_{b\lambda}(T_s) = \left(\frac{R_s}{d_{se}}\right)^2 \frac{C_1}{\lambda^5\left[\exp\left(\frac{C_2}{\lambda T_s}\right)-1\right]} \tag{2}$$

where R_s is the radius of the sun, d_{se} the distance between the sun and the earth, and C_1 and C_2 are constants.

(a) The total solar energy absorbed by the particle is

$$q_a = \int_0^\infty \frac{\pi d^2}{4} G_\lambda Q_a d\lambda \tag{3}$$

Plug Eqs. (1) and (2) into Eq. (3) to yield

$$q_a = \frac{0.133\pi^2 d^3}{4}\left(\frac{R_s}{d_{se}}\right)^2 T_s^5 \int_0^\infty \frac{C_1}{(\lambda T_s)^6\left[\exp\left(\frac{C_2}{\lambda T_s}\right)-1\right]} d(\lambda T_s) \tag{4}$$

Let $y = \frac{1}{\lambda T_s}$. Equation (4) becomes

$$q_a = \frac{0.133\pi^2 d^3}{4}\left(\frac{R_s}{d_{se}}\right)^2 C_1 T_s^5 \int_0^\infty \frac{y^{5-1}}{e^{C_2 y}-1} dy = \frac{0.133\pi^2 d^3}{4}\left(\frac{R_s}{d_{se}}\right)^2 \frac{C_1}{C_2^5}\Gamma(5)\zeta(5) T_s^5 \tag{5}$$

$(\Gamma(5) = 4! = 24$, *and for* $\nu \ge 5$, $\zeta(\ ) \approx 1.)$

(b) The scattering efficiency of the small particle is

$$Q_s = \frac{8}{3}x^4\left|\frac{m^2-1}{m^2+2}\right|^2 = \frac{8}{3}\left(\frac{\pi d}{\lambda}\right)^4\left|\frac{(2+0.1i)^2-1}{(2+0.1i)^2+2}\right|^2 = 0.673\left(\frac{\pi d}{\lambda}\right)^4 \tag{6}$$

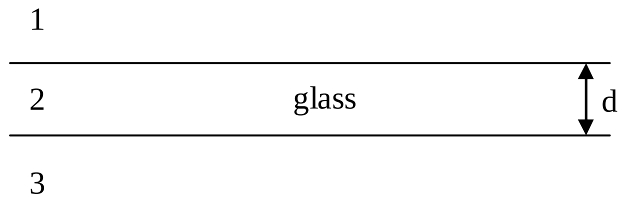
Similar to part (a), the scattered energy is given by

$$q_a = \frac{0.673\pi^5 d^6}{4}\left(\frac{R_s}{d_{se}}\right)^2 C_1 T_s^8 \int_0^\infty \frac{1}{(\lambda T_s)^9\left[\exp\left(\frac{C_2}{\lambda T_s}\right)-1\right]} d(\lambda T_s)$$

$$= \frac{0.673\pi^5 d^6}{4}\left(\frac{R_s}{d_{se}}\right)^2 \frac{C_1}{C_2^8}\Gamma(8)\zeta(8)T_s^8$$

Question 2.

(a)



The reflectivity at the interfaces is given by

$$\rho = \rho_{12} = \rho_{23} = |r_{12}|^2 = \left|\frac{N_2 - 1}{N_2 + 1}\right|^2 = \left|\frac{0.5 + i\kappa_2}{2.5 + i\kappa_2}\right|^2$$

(i) For $\lambda \le 8\mu m$, $\kappa_2 = 0$

$$\rho_{12} = 0.04$$

The peak wavelength of the radiation from the oven is around 1.5 μm, which is much less than the thickness of the glass. We can use ray tracing to calculate the reflectance of the glass slab.

$$R_{slab} = \rho\left[1 + \frac{(1-\rho)^2}{1-\rho^2}\right] = \frac{2\rho}{1+\rho} = 0.077$$

$$T_{slab} = \frac{1-\rho}{1+\rho} = 0.923$$

$$A_{slab} = 0$$

$$\varepsilon_\lambda = 0$$

(ii) For $\lambda > 8\mu m$, $\kappa_2 << 1$

$$\rho \approx 0.04$$

$$\tau = e^{-\kappa_2 d \cdot 4\pi/\lambda_0} = e^{-100\lambda_0 \frac{4\pi}{\lambda_0}\cdot d} = 0.285$$

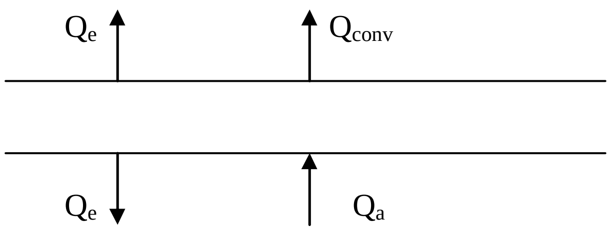
$$R_{slab} = \rho\left[1+\frac{(1-\rho)^2\tau^2}{1-\rho^2\tau^2}\right] = 0.043$$

$$T_{slab} = \frac{(1-\rho)^2\tau}{1-\rho^2\tau^2} = 0.262$$

$$A_{slab} = 1 - R_{slab} - T_{slab} = 0.695$$

$$\varepsilon_\lambda = 0.695$$

(b)



The glass view port absorbs irradiance from the oven and loses heat by radiation from both surfaces as well as convection to the air. Since the area of the oven surface is much larger than that of the glass, the oven can be treated as a blackbody source. The radiation absorbed by the bottom surface of the glass is then given by:

$$Q_a = A\alpha\sigma T_{oven}^4$$

where A is the area of the glass surface (one side), and α is the total hemispherical absorptance of the glass, which is given by

$$\alpha = \frac{\int_0^\infty \varepsilon_\lambda e_{b\lambda} d\lambda}{\sigma T_{oven}^4} = \frac{\int_{8\mu m}^\infty \varepsilon_\lambda e_{b\lambda} d\lambda}{\sigma T_{oven}^4} = 0.695\left(1 - f\left(8\mu m \cdot \ _{oven}\right)\right) = 0.018$$

where $f(\lambda T)$ can be found in the table of blackbody emissive power.
Similarly, the total hemispherical emissivity of the glass is given by

$$\varepsilon = 0.695\left(1 - f\left(8\mu m \cdot \ _{glass}\right)\right)$$

Apply energy balance to the glass view port:

$$\alpha\sigma T_{oven}^4 = 2\varepsilon T_{glass}^4 + h\left(\ _{glass} - T_\infty\right) = 1.39\left(1 - f\left(8\mu m \cdot \ _{glass}\right)\right)T_{glass}^4 + h\left(T_{glass} - T_\infty\right)$$

where we neglect the absorbed radiation from the low temperature environment. Plug in numbers and the above equation becomes

$$1.633\times10^4 = 1.39\left(1 - f\left(8\mu m\cdot T_{glass}\right)\right)T_{glass}^4 + 10\left(T_{glass} - 293\right)$$

(c) Iteration with the table value for f, we get, $T_{glass} \approx 930\text{K}$