18.435/2.111 Homework # 1

Due Thursday, September 18

Do two of the problems 2.29 – 2.32 in Nielsen and Chuang.

2.29: Show that the tensor product of two unitary operators is unitary.

2.30: Show that the tensor product of two Hermitian operators is Hermitian.

2.31: Show that the tensor product of two positive operators is positive.

2.32: Show that the tensor product of two projectors is a projector.

Do the problems 2.59 to 2.61 in Nielsen and Chuang. To do these, it is necessary to have read the preceding paragraph. Since the bookstore is out of books, I reproduce it here.

"Let's look at an example of projective measurements on single qubits. First is the measurement of the observable Z [This is their alternate notation for σ_z]. This has eigenvalues $+1$ and -1 with corresponding eigenvectors $|\,0\rangle$ and $|\,1\rangle$. Thus, for example, measurement of Z on the state $|\,\psi\rangle = (|\,0\rangle + |\,1\rangle)/\sqrt{2}$ gives the result $+1$ with probability $\langle\psi|0\rangle\langle 0|\psi\rangle = 1/2$, and similarly the result -1 with probability $1/2$. More generally, suppose $\vec{v}$ is any real three-dimensional unit vector. Then we can define an observable

$$\vec{v} \cdot \vec{\sigma} \equiv v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3.$$

Measurment of this observable is sometimes referred to as a 'measurement of spin along the $\vec{v}$ axis', for historical reasons. The following two exercises encourage you to work out some elementary but important properties of such a measurement."

2.59: Suppose we have a qubit in the state $|\,0\rangle$, and we measure the observable X [σ_x]. What is the average value of X? What is the standard deviation of X?

2.60: Show that $\vec{v}\cdot\vec{\sigma}$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_\pm \equiv (I \pm \vec{v} \cdot \vec{\sigma})/2$.

2.61: Calculate the probability of obtaining the result $+1$ for a measurement of $\vec{v} \cdot \vec{\sigma}$, given that the state prior to measurement is $|\,0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

The following three problems show a different proof that quantum mechanics is not a local realistic theory.

1: Consider the state $\frac{1}{\sqrt{2}}(|\,000\rangle + |\,111\rangle)$. Suppose all three qubits of this state are measured in the $\{\frac{1}{\sqrt{2}}(|\,0\rangle + |\,1\rangle), \frac{1}{\sqrt{2}}(|\,0\rangle - |\,1\rangle)\}$ basis. What are the possible joint outcomes of these measurements? With what probabilities do they occur? Suppose the first qubit is measured in the $\{\frac{1}{\sqrt{2}}(|\,0\rangle + |\,1\rangle), \frac{1}{\sqrt{2}}(|\,0\rangle - |\,1\rangle)\}$ basis and the second and third qubits in the $\{\frac{1}{\sqrt{2}}(|\,0\rangle + i\,|\,1\rangle), \frac{1}{\sqrt{2}}(|\,0\rangle - i\,|\,1\rangle)\}$ basis. Again, what are the possible joint outcomes of these measurements? With what probabilities do they occur? What is the expected value of the observable $\sigma_x(1) \otimes \sigma_x(2) \otimes \sigma_x(3)$? Of the observable $\sigma_x(1) \otimes \sigma_y(2) \otimes \sigma_y(3)$?

2: (Note: This problem by itself has no quantum mechanics in it.) Let f_1, f_2, f_3 be functions mapping the set $\{x, y\}$ to the set $\{1, -1\}$. Define

$$A_1 = f_1(x)f_2(x)f_3(x), \quad A_2 = f_1(y)f_2(y)f_3(x),$$

$$A_3 = f_1(y)f_2(x)f_3(y), \quad A_4 = f_1(x)f_2(y)f_3(y).$$

Show that either $A_1 = -1$ or one of A_2, A_3, A_4 equals $+1$.

3: Use the answers to problems 1 and 2 to show that quantum mechanics is not a local realistic theory.