Problem Set 1

Our objective in what follows is to introduce concepts of fixed points, stability, bifurcations, and chaos via experimental (i.e., numerical) computation. We choose a very simple mathematical model; although it is highly abstract, we will argue later in the course that the model is also to a significant degree representative of the behavior of many real systems.

Aside from serving as a kind of introduction to the course, this assignment also serves to introduce the class to the kind of computing problems we shall periodically assign. If you have any technical problems, please see us.

Let us parameterize a system by a single time-dependent variable, $x(t)$. The variable x could represent, say, the globally averaged temperature on the Earth's surface, the size of a particular population of animals on some secluded island, or perhaps even a particular stock market average. Whatever $x(t)$ represents, let us suppose that we are interested only in values x_j at discrete times $t = j\tau$, where τ is some suitably small interval of time. The evolution of x_j in time may then be written

$$x_{j+1} = R(x_j), \tag{1}$$

where R is some function that describes the dynamics. For any of the examples cited above (the climate, the stock market, etc.), it is obvious that the "true" R would involve complicated equations. Rather than going into that kind of detail, we consider below some simpler systems.

1. Consider first perhaps the most simple R, such that

   $$x_{j+1} = \mu x_j \tag{2}$$

   and $-\infty < \mu < \infty$ is a parameter. Without using the computer, what can you predict about the value of x_j given an initial condition x_0? Qualitatively, what is different about the cases $|\mu| < 1$, $|\mu| > 1$, and $|\mu| = 1$? What is the qualitative difference between the evolution with $\mu > 0$ and $\mu < 0$?

2. The Matlab function `iterate.m`[1] has been set up to solve equation (2). If you have any doubts concerning how to run the computer, or any doubts concerning your answers above, use the computer to verify your predictions for equation (2). (You need not hand in these plots.)

3. Although equation (2) could have real-world applications [e.g., μ could represent (1 + the rate of interest) for a monetary investment], its dynamics are not really that interesting. Consider instead the nonlinear evolution

   $$x_{j+1} = 4\mu x_j(1 - x_j) \tag{3}$$

[1] If needed, see also the introduction to Matlab.

Show that if $x_j = \bar{x}$, where

$$\bar{x} = \frac{4\mu - 1}{4\mu}, \tag{4}$$

then $x_{j+1} = x_j$. The points $\bar{x}$ are called *fixed points* of the dynamics.

4. The parameter μ in equation (3) determines whether the fixed points are *stable*. Later we will discuss at length how to determine stability theoretically. For now, however, let us investigate the question of stability experimentally. Modify your computer program so that it simulates the evolution of equation (3). Verify with a few tests that if $0.25 < \mu < 0.75$, then for any initial x_0 such that $0 < x_0 < 1$, the iterates x_j eventually converge to the value predicted by equation (4). These fixed points are called stable because, even when x_0 is chosen to be different from $\bar{x}$, the evolution eventually returns to $x_j = \bar{x}$.

5. Try the same experiment by choosing $\mu = 0.76$. What happens now? Is the fixed point predicted by equation (4) still stable? What is the long-term, asymptotic behavior?

6. The value $\mu = 0.75$ is called a *critical* value of μ because the long-term evolution of x_j qualitatively changes as μ increases to $\mu = 0.75$ from below. We call changes to the dynamics "qualitative" if, say, a fixed point changes from stable to unstable, or the period of oscillation changes from, say, 2 iterations to 4 iterations. These qualitative changes are called *bifurcations*. Can you find another critical value of μ between $\mu = 0.75$ and $\mu = 1.0$ by numerical experimentation? (Restrict the initial condition to $0 < x_0 < 1$.) What qualitative change occurs now?

7. Graph the evolution of equation (3) for the case $\mu = 0.89$, for some initial condition x_0. Then do the same for an initial condition $x_0 + \epsilon$, where ϵ is very small (say, less than 0.001). Is there any difference in the long-term behavior of x_j?

8. Repeat the exercise of Question 7 for the case $\mu = 0.91$. What happens now?