Elements of Continuum Elasticity

David M. Parks

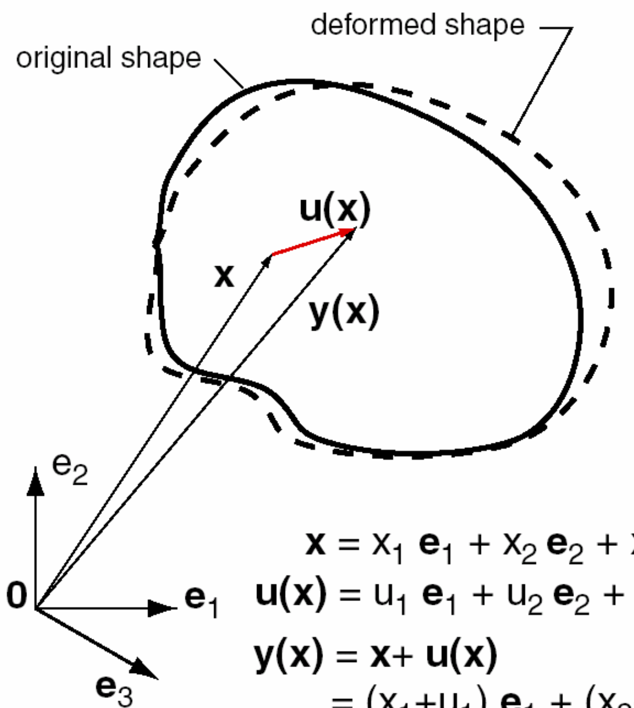
Mechanics and Materials II 2.002

February 25, 2004

Solid Mechanics in 3 Dimensions:
stress/equilibrium, strain/displacement, and intro to linear elastic constitutive relations

- **Geometry of Deformation**
  - –Position, 3 components of displacement, and [small] strain tensor
  - –Cartesian subscript notation; vectors and tensors
  - –Dilatation (volume change) and strain deviator
  - –Special cases: homogeneous strain; plane strain
- **Equilibrium of forces and moments:**
  - –Stress and ‘traction’
  - –Stress and equilibrium equations
  - –Principal stress; transformation of [stress] tensor components between rotated coordinate frames
  - –Special cases: homogeneous stress; plane stress
- **Constitutive connections: isotropic linear elasticity**
  - –Isotropic linear elastic material properties: E, ν, G, and K
  - –Stress/strain and strain/stress relations
  - –Putting it all together: Navier equations of equilibrium in terms of displacements
  - –Boundary conditions and boundary value problems

Geometry of Deformation



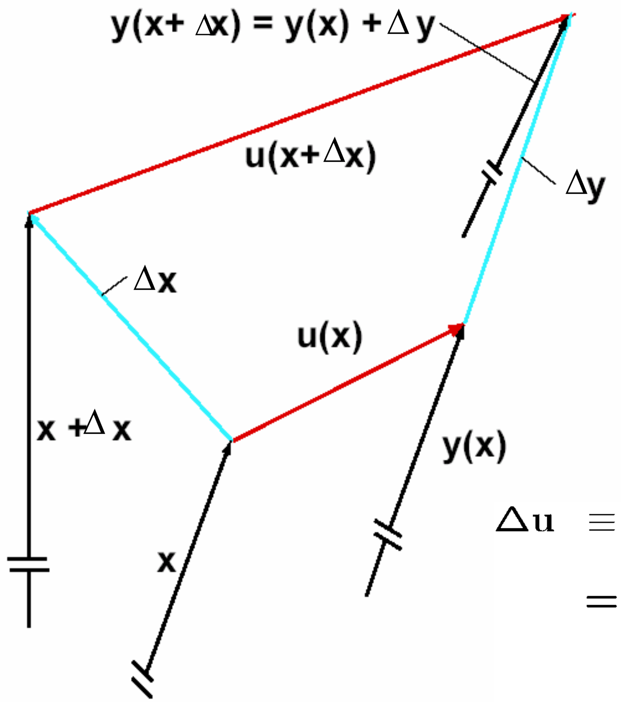
$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$$

$$\mathbf{u}(\mathbf{x}) = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$$

$$\mathbf{y}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) = (x_1+u_1)\,\mathbf{e}_1 + (x_2+u_2)\,\mathbf{e}_2 + (x_3+u_3)\,\mathbf{e}_3$$

- Origin : **0**; Cartesian basis vectors, $\mathbf{e}_1,\mathbf{e}_2$, & $\mathbf{e}_3$
- Reference location of material point : **x**; specified by its cartesian components, x_1, x_2, x_3
- Displacement vector of material point: **u(x)**; specified by displacement components, u_1, u_2, u_3
- Each function, u_i (I=1,2,3), in general depends on position **x** functionally through its components: e.g., $u_1 = u_1(x_1,x_2,x_3)$; etc.
- Deformed location of material point: **y(x)=x+u(x)**

Displacement of Nearby Points



- Neighboring points: **x** and **x** + Δ**x**
- Displacements: **u**(**x**) and **u**(**x** + Δ**x**)
- Deformed: **y**(**x**) and **y**(**x** + Δ**x**)
- Displacements: **u**(**x**) and **u**(**x** + Δ**x**)
- Vector geometry: Δ**y** = Δ**x** + Δ**u**, where Δ**u** = **u**(**x** + Δ**x**) - **u**(**x**)

$$
\begin{aligned}
\Delta \mathbf{u} &\equiv \mathbf{u}(\mathbf{x} + \Delta \mathbf{x}) - \mathbf{u}(\mathbf{x}) \\
&= \sum_{i=1}^{3} \mathbf{e}_i \left[u_i(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - u_i(x_1, x_2, x_3) \right] \\
&\equiv \sum_{i=1}^{3} \Delta u_i \, \mathbf{e}_i
\end{aligned}
$$

Displacement Gradient Tensor

Taylor series expansions of functions u_i:

$$\begin{aligned} u_i(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) &\doteq +u_i(x_1, x_2, x_3) \\ &\quad + \frac{\partial u_i}{\partial x_1}\Delta x_1 + \frac{\partial u_i}{\partial x_2}\Delta x_2 + \frac{\partial u_i}{\partial x_3}\Delta x_3 \\ &\quad + o(\Delta \mathrm{x}) \\ &= u_i(x_1, x_2, x_3) + \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\Delta x_j \end{aligned}$$

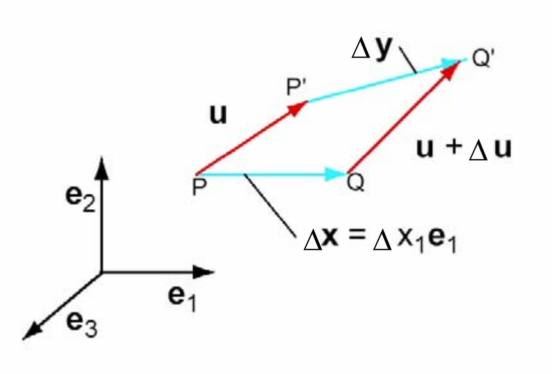
Thus, on returning to the expression on previous the slide, Δu_i is given, for each component (i=1,..3), by

$$\Delta u_i = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\Delta x_j$$

Components of the <u>displacement gradient tensor</u> can be put in matrix form:

$$\left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Displacement Gradient and Extensional Strain in Coordinate Directions



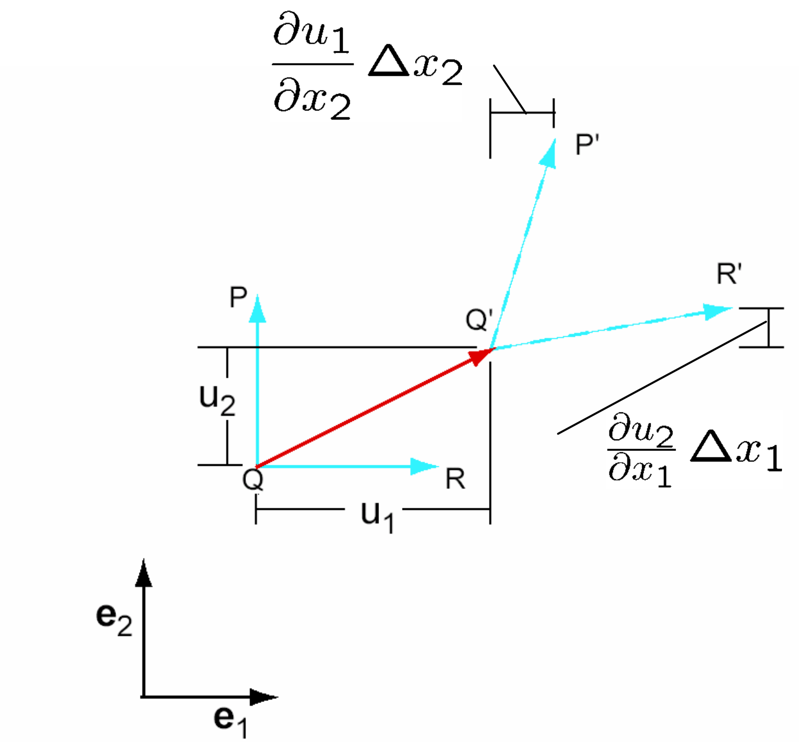
Suppose that $\Delta\mathbf{x} = \Delta x_1\,\mathbf{e}_1$;
Then, with $\Delta\mathbf{y} = \Delta\mathbf{x} + \Delta\mathbf{u}$,

$$
\begin{aligned}
\Delta \mathrm{y} &= \underbrace{\Delta x_1\,\mathrm{e}_1}_{\Delta \mathrm{x}} + \underbrace{\frac{\partial u_1}{\partial x_1}\Delta x_1\,\mathrm{e}_1 + \frac{\partial u_2}{\partial x_1}\Delta x_1\,\mathrm{e}_2 + \frac{\partial u_3}{\partial x_1}\Delta x_1\,\mathrm{e}_3}_{\Delta \mathrm{u}} \\
&= \Delta x_1 \left[(1+\frac{\partial u_1}{\partial x_1})\mathrm{e}_1 + \frac{\partial u_2}{\partial x_1}\mathrm{e}_2 + \frac{\partial u_3}{\partial x_1}\mathrm{e}_3\right]; \\
|\Delta \mathrm{y}| &= \sqrt{\Delta \mathrm{y}\cdot\Delta \mathrm{y}} \\
&= |\Delta x_1|\sqrt{(1+\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_1})^2} \\
&= |\Delta x_1|\sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \underbrace{[(\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_1})^2]}_{\text{higher-order}}} \\
&\doteq |\Delta x_1|\left[1+\frac{\partial u_1}{\partial x_1}\right] \Rightarrow
\end{aligned}
$$

$$
\boxed{\frac{|\Delta \mathrm{y}| - |\Delta \mathrm{x}|}{|\Delta \mathrm{x}|} = \frac{\partial u_1}{\partial x_1}}
$$

The <u>fractional change in length</u> (extensional strain) of a material line element initially parallel to x1 axis is $\partial u_1 / \partial x_1$; similar conclusions apply for coordinate directions 2 and 3

Displacement Gradient and Shear Strain



- Let QR = Δx_1 $\mathbf{e}_1$ & QP = $\Delta x_2 \mathbf{e}_2$
- Line segments initially perpendicular
- Deformed lines: Q'R' & Q'P'
- $|Q'R'| = |\Delta x_1|(1 + \partial u_1/\partial x_1)$
- $|Q'P'| = |\Delta x_2|(1 + \partial u_2/\partial x_2)$

$$\angle P'Q'R' = \pi/2 - (\theta_1 + \theta_2)$$

$$\sin\theta_1 = \frac{\frac{\partial u_2}{\partial x_1}\Delta x_1}{|Q'R'|} = \frac{\frac{\partial u_2}{\partial x_1}}{(1 + \frac{\partial u_1}{\partial x_1})} \Rightarrow$$

$$\sin\theta_1 \doteq \theta_1 \doteq \frac{\partial u_2}{\partial x_1}; \text{similarly}$$

$$\sin\theta_2 \doteq \theta_2 \doteq \frac{\partial u_1}{\partial x_2}$$

The total reduction in angle of 2 line segments initially perpendicular to coordinate axes 1 and 2 is

$$\theta_1 + \theta_2 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

Similar results apply for all axis pairs

Strain Tensor (I)

The cartesian components of the [small] strain tensor are given, for i=1..3 and j=1..3, by

$$\epsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Written out in matrix notation, this index equation is

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

- Each of the 9 components in the 3×3 matrices on each side of the matrix equation are equal, so this is equivalent to 9 separate equations.
- The strain tensor is symmetric, in that, for each i and j, $\varepsilon_{ij} = \varepsilon_{j;i}$

Strain Tensor (II)

The cartesian components of the [small] strain tensor are given, for i=1..3 and j=1..3, by

$$\epsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Written out in matrix notation, this index equation is

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

- Diagonal components of the strain tensor are the extensional strains along the respective coordinate axes;
- Off-diagonal components of the strain tensor are ½ times the total reduction in angle (from $\pi/2$) of a pair of deformed line elements that were initially parallel to the two axes indicated by the off-diagonal row and column number

Fractional Volumetric Change

For <u>any</u> values of the strain tensor components, ε_{ij},
the fractional volume change at a material point, sometimes called the <u>dilatation</u> at the point, is given by

$$\frac{V_{\text{deformed}} - V_{\text{initial}}}{V_{\text{initial}}} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

$$= \sum_{k=1}^{3} \epsilon_{kk}$$

This relation holds whether or not the values of ε_{11}, ε_{22}, and ε_{33} equal each other, and whether or not any or all of the shear strain components (e.g., $\varepsilon_{12}=\varepsilon_{21}$) are zero-valued or non-zero-valued.

The sum of diagonal elements of a matrix of the cartesian components of a tensor is called the <u>trace</u> of the tensor; thus,
the fractional volume change is the trace of the strain tensor.

Strain Deviator Tensor

Components of the strain deviator tensor, are given in terms of the components of the strain tensor by

$$\epsilon_{ij}^{(\text{dev})} \equiv \epsilon_{ij} - \frac{1}{3}\delta_{ij} \sum_{k=1}^{3} \epsilon_{kk}$$

$$\left[\delta_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here δ_{ij} are components of the Kronecker identity matrix, satisfying δ_{ij} =1 if i=j, and δ_{ij} =0 if i is not equal to j

- Off-diagonal components of the strain deviator tensor equal corresponding off-diagonal components of the strain tensor;
- Each diagonal component of the strain deviator tensor differs from the corresponding diagonal component of the strain tensor by 1/3 of the trace of the strain tensor

Exercise: evaluate the trace of the strain deviator tensor.

Strain Decomposition

Alternatively, the strain tensor can be viewed as the sum of

•a shape-changing (but volume-preserving) part (the strain deviator)
Plus
•a volume-changing (but shape-preserving) part (one-third trace of strain tensor times identity matrix):

$$\epsilon_{ij} = \underbrace{\epsilon_{ij}^{(\text{dev})}}_{\text{shape-changing}} + \underbrace{\frac{1}{3}\delta_{ij}\sum_{k=1}^{3}\epsilon_{kk}}_{\text{volume-changing}}$$

Later, when we look more closely at isotropic linear elasticity, we will find that the two "fundamental" elastic constants are

•the <u>bulk modulus</u>, K, measuring elastic resistance to volume-change, and
• the <u>shear modulus,</u> G, measuring elastic resistance to shape-change

Geometric Aspects of Strain

Undeformed segment:

Δ**x:** <u>undeformed</u> vector from P to Q
Δs : length of vector = |PQ|
e$_{(P\text{->}Q)}$: unit vector pointing in direction from P to Q

$$\Delta \mathbf{x} = \Delta s \, \mathbf{e}_{(P\to Q)}$$

$$\Delta s = |\Delta \mathbf{x}| = \sqrt{\Delta \mathbf{x} \cdot \Delta \mathbf{x}}$$

$$\mathbf{e}_{(P\to Q)} \equiv \frac{\Delta \mathbf{x}}{\Delta s}$$

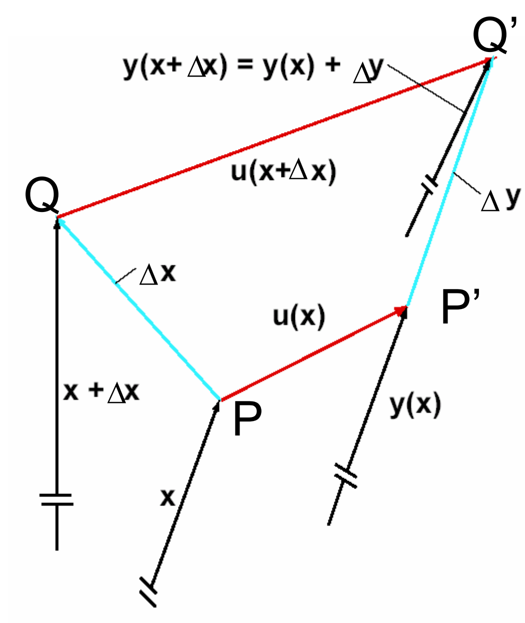
Deformed segment:

Δ**y:** <u>deformed</u> vector from P' to Q'
ΔS : length of vector = |P'Q'|
e$_{(P'\text{->}Q')}$: unit vector pointing in direction from P' to Q'

$$\Delta \mathbf{y} = \Delta S \, \mathbf{e}_{(P'\to Q')}$$

$$\Delta S = |\Delta \mathbf{y}| = \sqrt{\Delta \mathbf{y} \cdot \Delta \mathbf{y}}$$

$$\mathbf{e}_{(P'\to Q')} \equiv \frac{\Delta \mathbf{y}}{\Delta S}$$



Fractional Length Change: Arbitrary Initial Direction

Undeformed:

$$\Delta \mathbf{x} = \Delta s\, \mathbf{e}_{(P\to Q)}$$

$$\Delta s = |\Delta \mathbf{x}| = \sqrt{\Delta \mathbf{x}\cdot\Delta \mathbf{x}}$$

$$\mathbf{m} \equiv \mathbf{e}_{(P\to Q)} = \frac{\Delta \mathbf{x}}{\Delta s}$$

$$\mathbf{m} = \sum_{i=1}^{3} m_i\,\mathbf{e}_i;$$

$$m_i = \mathbf{m}\cdot\mathbf{e}_i;$$

$$1 = \mathbf{m}\cdot\mathbf{m} = m_1^2 + m_2^2 + m_3^2$$

$$\Delta \mathbf{x} = \Delta s\,\mathbf{m} \Longleftrightarrow \Delta x_i = \Delta s\, m_i$$

Deformed length (squared):

$$\begin{aligned}(\Delta S)^2 = |\Delta \mathbf{y}|^2 &= \Delta \mathbf{y}\cdot\Delta\mathbf{y}\\ &= (\Delta\mathbf{x}+\Delta\mathbf{u})\cdot(\Delta\mathbf{x}+\Delta\mathbf{u})\\ &= \underbrace{\Delta\mathbf{x}\cdot\Delta\mathbf{x}}_{(\Delta s)^2} + \Delta\mathbf{x}\cdot\Delta\mathbf{u} + \Delta\mathbf{u}\cdot\Delta\mathbf{x} + \Delta\mathbf{u}\cdot\Delta\mathbf{u}\\ &= (\Delta s)^2 + \Delta s(\mathbf{m}\cdot\Delta\mathbf{u} + \Delta\mathbf{u}\cdot\mathbf{m}) + \Delta\mathbf{u}\cdot\Delta\mathbf{u}\\ &= (\Delta s)^2 + (\Delta s)\left(\sum_{i=1}^{3} m_i\,\Delta u_i + \sum_{j=1}^{3}\Delta u_j m_j\right) + \sum_{i=1}^{3}\Delta u_i \Delta u_i\end{aligned}$$

But,

$$\begin{aligned}\Delta u_i &= \sum_{j=1}^{3}\frac{\partial u_i}{\partial x_j}\Delta x\\ &= \Delta s \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_j}\end{aligned}$$

Same idea,
But sum on i

Finally:

$$\Delta S = \Delta s\sqrt{1 + \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) m_i m_j};$$

$$\frac{|\Delta\mathbf{y}| - |\Delta\mathbf{x}|}{|\Delta\mathbf{x}|} = \frac{\Delta S - \Delta s}{\Delta s} = \frac{1}{2}\sum_{i=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) m_i m_j$$

The fractional change in length for a line element initially parallel to ANY unit vector $\pm$ **m** is given in terms of direction cosines, m_i, and the displacement gradient components by

Local Axial Strain in Any Direction

Strain along unit direction **m**:

$$\frac{|\Delta \mathbf{y}| - |\Delta \mathbf{x}|}{|\Delta \mathbf{x}|} = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) m_i m_j$$

$$\epsilon_{\mathbf{m}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ij}\, m_i m_j$$

Vector components of **m**:

$$\{m_i\} = \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \end{Bmatrix} (3 \times 1)$$

$$\lfloor m_i \rfloor = \lfloor \; m_1 \quad m_2 \quad m_3 \; \rfloor \, (1 \times 3)$$

[Extended] matrix multiplication provides strain in direction parallel to **m**:

$$\epsilon_{\mathbf{m}} = \underbrace{\lfloor \; m_1 \quad m_2 \quad m_3 \; \rfloor}_{1\times 3} \underbrace{\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}}_{3\times 3} \underbrace{\begin{Bmatrix} m_1 \\ m_2 \\ m_3 \end{Bmatrix}}_{3\times 1}$$

Example

Suppose that the components of the strain tensor are

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} 0.003 & -0.001 & 0.002 \\ -0.001 & -0.002 & 0. \\ 0.002 & 0. & -0.002 \end{bmatrix}$$

Find the fractional change in length of a line element initially pointing
Along the direction **m** = ($\mathbf{e}_1$ + $\mathbf{e}_2$ + $\mathbf{e}_3$) / $3^{1/2}$

Solution: equal components $m_i = 1 / (3)^{1/2}$

$$\begin{aligned} \epsilon_{\mathrm{m}} &= \lfloor\, 1/\sqrt{3} \;\; 1/\sqrt{3} \;\; 1/\sqrt{3} \,\rfloor \begin{bmatrix} 0.003 & -0.001 & 0.002 \\ -0.001 & -0.002 & 0. \\ 0.002 & 0. & -0.002 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} \\ &= \frac{1}{3} \times 0.001 = 0.000333 \end{aligned}$$

Change of Basis Vectors; Change of Components: but No Change in Vector

Given:

- a vector **v**;
- 2 sets of cartesian basis vectors:
  { $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$} and {$\mathbf{e}_1$', $\mathbf{e}_2$', $\mathbf{e}_3$'}

- components of **v** wrt {$\mathbf{e}_i$}: {v_i} ;
- components of **v** wrt {$\mathbf{e}_i$'}: {v_i'} ;

$$\mathbf{v} = \sum_{i=1}^{3} v_i\, \mathrm{e}_i = \sum_{j=1}^{3} v'_j\, \mathrm{e}'_j$$

Question: what relationships exist connecting
The components of **v** in the two bases?

Vector Dot Product and Vector Components

Consider the following dot product operations:

$$\mathbf{e}_1 \cdot \mathbf{v} = \mathbf{e}_1 \cdot (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) = v_1$$

$$\mathbf{e}'_2 \cdot \mathbf{v} = \mathbf{e}'_2 \cdot (v'_1\mathbf{e}'_1 + v'_2\mathbf{e}'_2 + v'_3\mathbf{e}'_3) = v'_2$$

Evidently, for <u>any</u> basis vector (primed or unprimed)

$$v_i = \mathbf{e}_i \cdot \mathbf{v}$$

$$v'_j = \mathbf{e}'_j \cdot \mathbf{v}$$

Thus, <u>any vector</u> **v** can be expressed as

$$\mathbf{v} = \sum_{i=1}^{3} v_i\mathbf{e}_i = \sum_{i=1}^{3} (\mathbf{v} \cdot \mathbf{e}_i)\mathbf{e}_i$$

$$\mathbf{v} = \sum_{j=1}^{3} v'_j\mathbf{e}'_j = \sum_{j=1}^{3} (\mathbf{v} \cdot \mathbf{e}'_j)\mathbf{e}'_j$$

Changing Coordinate Systems (I)

Define a matrix Q_{ij} by $$Q_{ij} \equiv \mathbf{e}'_i \cdot \mathbf{e}_j$$

$$[Q_{ij}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{e}'_1 \cdot \mathbf{e}_1 & \mathbf{e}'_1 \cdot \mathbf{e}_2 & \mathbf{e}'_1 \cdot \mathbf{e}_3 \\ \mathbf{e}'_2 \cdot \mathbf{e}_1 & \mathbf{e}'_2 \cdot \mathbf{e}_2 & \mathbf{e}'_2 \cdot \mathbf{e}_3 \\ \mathbf{e}'_3 \cdot \mathbf{e}_1 & \mathbf{e}'_3 \cdot \mathbf{e}_2 & \mathbf{e}'_3 \cdot \mathbf{e}_3 \end{bmatrix}$$

Express primed components in terms of unprimed:

$$v'_i = \mathbf{e}'_i \cdot \mathbf{v} = \mathbf{e}'_i \cdot \left(\sum_{j=1}^{3} v_j \mathbf{e}_j\right) = \sum_{j=1}^{3} Q_{ij} v_j$$

$$\{v'_i\} = [Q_{ij}]\{v_j\}$$

$$\{v_i\} = [Q_{ij}]^T \{v'_j\}$$

Alternatively, matrix multiplication to convert vector components:

$$\begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Note: the matrix $[Q_{ij}]$ is said to be <u>orthogonal</u>:

- Determinant of $[Q_{ij}] = 1$
- Matrix transpose is matrix inverse: $[Q_{ij}]^{-1} = [Q_{ij}]^T = [Q_{ji}]$

Changing Coordinate Systems (II)

Define a matrix Q_{ij} by

$$[Q_{ij}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{e}'_1 \cdot \mathbf{e}_1 & \mathbf{e}'_1 \cdot \mathbf{e}_2 & \mathbf{e}'_1 \cdot \mathbf{e}_3 \\ \mathbf{e}'_2 \cdot \mathbf{e}_1 & \mathbf{e}'_2 \cdot \mathbf{e}_2 & \mathbf{e}'_2 \cdot \mathbf{e}_3 \\ \mathbf{e}'_3 \cdot \mathbf{e}_1 & \mathbf{e}'_3 \cdot \mathbf{e}_2 & \mathbf{e}'_3 \cdot \mathbf{e}_3 \end{bmatrix}$$

Express unprimed components in terms of primed:

$$v_i = \mathbf{e}_i \cdot \mathbf{v} = \mathbf{e}_i \cdot (\sum_{j=1}^{3} v'_j \mathbf{e}'_j) = \sum_{j=1}^{3} Q_{ji} v'_j$$

$$\{v'_i\} = [Q_{ij}]\{v_j\}$$

$$\{v_i\} = [Q_{ij}]^T \{v'_j\}$$

Matrix multiplication to convert vector components:

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix}$$

$$= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}^T \begin{Bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{Bmatrix}$$

Note: the matrix $[Q_{ij}]$ is said to be orthogonal:

- Determinant of $[Q_{ij}] = 1$
- Matrix transpose is matrix inverse: $[Q_{ij}]^{-1} = [Q_{ij}]^T = [Q_{ji}]$

Transformation of Displacement Gradient (Tensor) Components

Vector/vector operation (unprimed components):

$$\{\Delta u_i\} = \left[\frac{\partial u_i}{\partial x_j}\right]\{\Delta x_j\}$$

Pre-multiply by [Q]:

$$\underbrace{[Q_{mi}]\{\Delta u_i\}}_{\{\Delta u'_m\}} = [Q_{mi}]\left[\frac{\partial u_i}{\partial x_j}\right]\underbrace{\{\Delta x_j\}}_{[Q_{jn}]^T\{\Delta x'_n\}}$$

Substitute on both sides:

$$\{\Delta u'_m\} = \underbrace{[Q_{mi}]\left[\frac{\partial u_i}{\partial x_j}\right]\left[Q_{jn}\right]^T}_{[\partial u'_m/\partial x'_n]}\{\Delta x'_n\}$$

Vector/vector operation in primed components:

$$\{\Delta u'_m\} = \left[\frac{\partial u'_m}{\partial x'_n}\right]\{\Delta x'_n\}$$

This must always hold so that

$$\left[\frac{\partial u'_m}{\partial x'_n}\right] = [Q_{mi}]\left[\frac{\partial u_i}{\partial x_j}\right]\left[Q_{jn}\right]^T$$

This procedure transforms the cartesian components of any second-order tensor, including ε_{ij}

Change of Tensor Components with Respect to Change of Basis Vectors

For each primed index, i' and j', the tensor component with respect to the primed basis vectors, $A_{i'j'}$, is given by

$$A_{i'j'} = \sum_{m=1}^{3} \sum_{n=1}^{3} Q_{i'm} Q_{j'n} A_{mn}$$

Alternatively, the complete matrix of the primed components of the tensor can be obtained from matrix multiplication:

$$\begin{bmatrix} A_{1'1'} & A_{1'2'} & A_{1'3'} \\ A_{2'1'} & A_{2'2'} & A_{2'3'} \\ A_{3'1'} & A_{3'2'} & A_{3'3'} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix}^T$$

for any second-order tensor **A**