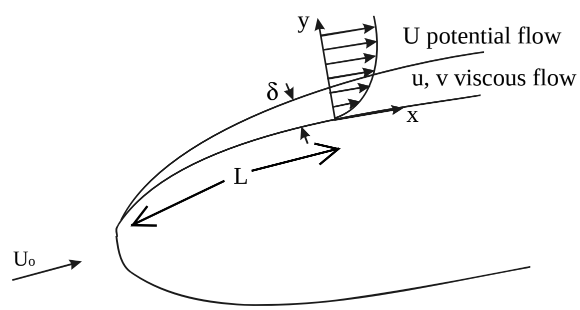
2.20 - Marine Hydrodynamics
Lecture 17

4.6 Laminar Boundary Layers



4.6.1 Assumptions

- 2D flow: $w, \dfrac{\partial}{\partial z} \equiv 0$ and $u(x,y), v(x,y), p(x,y), U(x,y)$.
- Steady flow: $\dfrac{\partial}{\partial t} \equiv 0$.
- For $\delta << L$, use local (body) coordinates x, y, with x tangential to the body and y normal to the body.
- $u \equiv$ tangential and $v \equiv$ normal to the body, viscous flow velocities (used inside the boundary layer).
- $U, V \equiv$ potential flow velocities (used outside the boundary layer).

4.6.2 Governing Equations

- Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

- Navier-Stokes:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

4.6.3 Boundary Conditions

- *KBC*

  Inside the boundary layer:

$$\text{No-slip } u(x, y=0) = 0$$

$$\text{No-flux } v(x, y=0) = 0$$

  Outside the boundary layer the velocity has to match the P-Flow solution. Let $y^\star \equiv y/\delta$, $y^* \equiv y/L$, and $x^* \equiv x/L$. Outside the boundary layer $y^\star \to \infty$ but $y^* \to 0$. We can write for the tangential and normal velocities
  $u(x^*, y^\star \to \infty) = U(x^*, y^* \to 0) \Rightarrow u(x^*, y^\star \to \infty) = U(x^*, 0)$,
  and $v(x^*, y^\star \to \infty) = V(x^*, y^* \to 0) \Rightarrow v(x^*, y^\star \to \infty) = V(x^*, 0) \underset{\substack{\text{No-flux} \\ \text{P-Flow}}}{=} 0$

  In short:

$$u(x, y^\star \to \infty) = U(x, 0)$$

$$v(x, y^\star \to \infty) = 0$$

- *DBC*

  As $y^\star \to \infty$, the pressure has to match the P-Flow solution. The x-momentum equation at $y^* = 0$ gives

$$U\frac{\partial U}{\partial x} + \underset{\substack{\downarrow \\ 0}}{V}\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \underset{\substack{\downarrow \\ 0}}{\nu}\frac{\partial^2 U}{\partial y^2} \Rightarrow \frac{dp}{dx} = -\rho U\frac{\partial U}{\partial x}$$

4.6.4 Boundary Layer Approximation

Assume that $R_{e_L} >> 1$, then (u, v) is confined to a thin layer of thickness $\delta(x) << L$. For flows within this boundary layer, the appropriate order-of-magnitude scaling / normalization is:

Variable	Scale	Normalization
u	$\mathcal{U}$	$u = \mathcal{U}u^*$
x	L	$x = Lx^*$
y	δ	$y = \delta y^*$
v	$\mathcal{V} =?$	$v = \mathcal{V}v^*$

- Non-dimensionalize the continuity, Equation (1), to relate $\mathcal{V}$ to $\mathcal{U}$

$$\frac{\mathcal{U}}{L}\left(\frac{\partial u}{\partial x}\right)^* + \frac{\mathcal{V}}{\delta}\left(\frac{\partial v}{\partial y}\right)^* = 0 \Longrightarrow \mathcal{V} = \mathrm{O}\left(\frac{\delta}{\mathrm{L}}\mathcal{U}\right)$$

- Non-dimensionalize the x-momentum, Equation (2), to compare δ with L

$$\frac{\mathcal{U}^2}{L}\left(u\frac{\partial u}{\partial x}\right)^* + \underbrace{\frac{\mathcal{U}\mathcal{V}}{\delta}}_{O(\mathcal{U}^2/L)}\left(v\frac{\partial u}{\partial y}\right)^* = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu\mathcal{U}}{\delta^2}\left[\underbrace{\frac{\delta^2}{L^2}\left(\frac{\partial^2 u}{\partial x^2}\right)^*}_{ignore} + \left(\frac{\partial^2 u}{\partial y^2}\right)^*\right]$$

The inertial effects are of comparable magnitude to the viscous effects when:

$$\frac{\mathcal{U}^2}{L} \sim \frac{\nu\mathcal{U}}{\delta^2} \Longrightarrow \frac{\delta}{L} \sim \sqrt{\frac{\nu}{\mathcal{U}L}} = \frac{1}{R_{e_L}} << 1$$

The pressure gradient $\frac{\partial p}{\partial x}$ must be of comparable magnitude to the inertial effects

$$\frac{\partial p}{\partial x} = O\left(\rho\frac{\mathcal{U}^2}{L}\right)$$

- Non-dimensionalize the y-momentum, Equation (3), to compare $\frac{\partial p}{\partial y}$ to $\frac{\partial p}{\partial x}$

$$\underbrace{\frac{\mathcal{U}\mathcal{V}}{L}}_{O(\frac{\mathcal{U}^2}{L}\frac{\delta}{L})}\left(u\frac{\partial v}{\partial x}\right)^* + \underbrace{\frac{\mathcal{V}^2}{\delta}}_{O(\frac{\mathcal{U}^2}{L}\frac{\delta}{L})}\left(v\frac{\partial v}{\partial y}\right)^* = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \underbrace{\frac{\nu\mathcal{V}}{L^2}}_{O(\frac{\mathcal{U}^2}{L}\frac{\delta^3}{L^3})}\left(\frac{\partial^2 v}{\partial x^2}\right)^* + \underbrace{\frac{\nu\mathcal{V}}{\delta^2}}_{O(\frac{\mathcal{U}^2}{L}\frac{\delta}{L})}\left(\frac{\partial^2 v}{\partial y^2}\right)^*$$

The pressure gradient $\frac{\partial p}{\partial y}$ must be of comparable magnitude to the inertial effects

$$\frac{\partial p}{\partial y} = O\left(\rho\frac{\mathcal{U}^2}{L}\frac{\delta}{L}\right)$$

Comparing the magnitude of $\frac{\partial p}{\partial x}$ to $\frac{\partial p}{\partial y}$ we observe

$$\begin{aligned} \frac{\partial p}{\partial y} &= O\left(\rho\frac{\mathcal{U}^2}{L}\frac{\delta}{L}\right) \text{ while } \frac{\partial p}{\partial x} = O\left(\rho\frac{\mathcal{U}^2}{L}\right) \Longrightarrow \\ \frac{\partial p}{\partial y} &<< \frac{\partial p}{\partial x} \qquad \Longrightarrow \frac{\partial p}{\partial y} \approx 0 \qquad \Longrightarrow \\ p &= p(x) \end{aligned}$$

- Note:
  - From continuity it was shown that $\mathcal{V}/\mathcal{U} \sim O(\delta/L) \Rightarrow v << u$, inside the boundary layer.
  - It was shown that $\frac{\partial p}{\partial y} = 0$, $p = p(x)$ inside the boundary layer. This means that the pressure across the boundary layer is constant and equal to the pressure **outside** the boundary layer imposed by the external P-Flow.

4.6.5 Summary of Dimensional BVP

Governing equations for 2D, steady, laminar boundary layer

$$\begin{aligned}
\text{Continuity} \quad &: \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
x\text{-momentum} \quad &: \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{dp}{dx}}_{UdU/dx,\ y=0} + \nu\frac{\partial^2 u}{\partial y^2} \\
y\text{ - momentum} \quad &: \quad \frac{\partial p}{\partial y} = 0
\end{aligned}$$

Boundary Conditions

KBC

$$\begin{aligned}
\text{At y=0} \quad &: \quad u(x,0) = 0 \\
&\quad\;\; v(x,0) = 0 \\
\text{At } y/\delta \to \infty \quad &: \quad u(x, y/\delta \to \infty) = U(x,0) \\
&\quad\;\; v(x, y/\delta \to \infty) = 0
\end{aligned}$$

DBC

$$\frac{dp}{dx} = -\rho U \frac{\partial U}{\partial x} \text{ or } \underbrace{\overset{\text{IN the b.l.}}{p(x)} = C - \frac{1}{2}\rho U^2(x,0)}_{\text{Bernoulli for the P-Flow at } y=0}$$

4.6.6 Definitions

Displacement thickness $\boxed{\delta^* \equiv \int_0^\infty \left(1 - \frac{u}{U}\right) dy}$

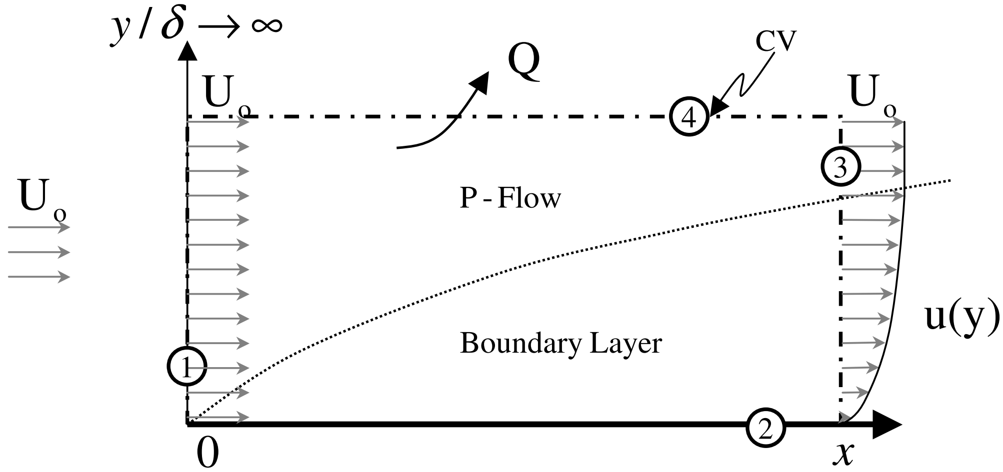
Momentum thickness $\boxed{\theta \equiv \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy}$

Physical Meaning of δ^* and θ

Assume a 2D steady flow over a flat plate.

Recall for steady flow over flat plate $\frac{dp}{dx} = 0$ and pressure $p = const.$

Choose a control volume ($[0, x] \times [0, y/\delta \to \infty]$) as shown in the figure below.



CV for steady flow over a flat plate.

Control Volume 'book-keeping'

Surface	$\hat{n}$	$\vec{v}$	$\vec{v} \cdot \hat{n}$	$\vec{v}(\vec{v} \cdot \hat{n})$	$-p\hat{n}$
①	$-\hat{i}$	$U_o\hat{i}$	$-U_o$	$-U_o^2\hat{i}$	$p\hat{i}$
②	$-\hat{j}$	0	0	0	$p\hat{j}$
③	$\hat{i}$	$u(x,y)\hat{i} + v(x,y)\hat{j}$	$u(x,y)$	$u^2(x,y)\hat{i} + u(x,y)v(x,y)\hat{j}$	$-p\hat{i}$
④	$\hat{j}$	$U_o\hat{i} + v(x,y)\hat{j}$	$v(x,y)$	$v(x,y)U_o\hat{i} + v^2(x,y)\hat{j}$	$-p\hat{j}$

Conservation of mass, for steady CV

$$\oint_{1234} \vec{v} \cdot \hat{n} dS = 0 \Rightarrow -\int_0^\infty U_o dy' + \int_0^\infty u(x,y')dy' + \underbrace{\int_0^x v(x',y)dx'}_{Q} = 0 \Rightarrow$$

$$Q = \int_0^\infty U_o dy' - \int_0^\infty u dy' = \int_0^\infty (U_o - u)dy' = U_o \underbrace{\int_0^\infty \left(1 - \frac{u}{U_o}\right)dy'}_{\delta^*} \Rightarrow \boxed{Q = U_0 \delta^*}$$

where $()'$ are the dummy variables.

Conservation of momentum in x, for steady CV

$$\oint_{1234} u(\vec{v} \cdot \hat{n})dS = \sum F_x \Rightarrow$$

$$\int_0^\infty -U_o^2 dy' + \int_0^\infty u^2(x,y')dy' + \int_0^x v(x',y)U_o dx' = \int_0^\infty p dy' - \int_0^\infty p dy' + \sum F_{x,friction} \Rightarrow$$

$$\int_0^\infty -U_o^2 dy' + \int_0^\infty u^2(x,y')dy' + U_o \underbrace{\int_0^x v(x',y)dx'}_{Q} = \sum F_{x,friction} \Rightarrow$$

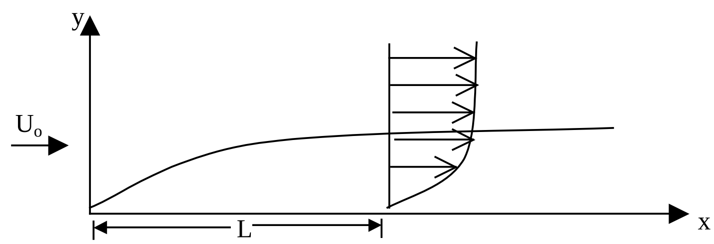
$$\int_0^\infty -U_o^2 dy' + \int_0^\infty u^2(x,y')dy' + U_o \int_0^\infty (U_o - u)dy' = \sum F_{x,friction} \Rightarrow$$

$$\int_0^\infty \Big(-U_o^2 + u^2 + U_o^2 - U_o u \Big) dy' = \sum F_{x,friction} \Rightarrow$$

$$U_o^2 \int_0^\infty \left(\frac{u^2}{U_o^2} - \frac{u}{U_o} \right) dy' = \sum F_{x,friction} \Rightarrow$$

$$\sum F_{x,friction} = -U_o^2 \underbrace{\int_0^\infty \frac{u}{U_o}\left(1 - \frac{u}{U_o}\right)dy'}_{\theta} \Rightarrow \boxed{\sum F_{x,friction} = -U_o^2 \theta}$$

4.7 Steady Flow over a Flat Plate: Blasius' Laminar Boundary Layer



Steady flow over a flat plate: BLBL

4.7.1 Derivation of BLBL

- *Assumptions* Steady, 2D flow. Flow over flat plate $\rightarrow U = U_0, V = 0, \dfrac{dp}{dx} = 0$
- *LBL governing equations*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

- *Boundary conditions*

$$u = v = 0 \text{ on } y = 0$$

$$v \rightarrow V = 0, u \rightarrow U_o \text{ outside the BL, i.e., } \left(\frac{y}{\delta} >> 1\right)$$

- *Solution* Mathematical solution in terms of similarity parameters.

$$\frac{u}{U} \quad \text{and} \quad \eta \equiv y\sqrt{\frac{U_o}{\nu x}} \Leftrightarrow \frac{y}{x} = \frac{\eta}{\sqrt{R_x}} \Leftrightarrow y = \eta\sqrt{\frac{\nu x}{U_o}}$$

Similarity solution must have the form

$$\underbrace{\frac{u\left(x,y\right)}{U_o} = F\left(\eta\right)}_{\text{self similar solution}}$$

We can obtain a PDE for F by substituting into the governing equations. The PDE has no-known analytical solution. However, Blasius provided a numerical solution. Once again, once the velocity profile is evaluated we know everything about the flow.

4.7.2 Summary of BLBL Properties: $\delta, \delta_{0.99}, \delta^*, \theta, \tau_o, D, C_f$

$$\frac{u(x,y)}{U_o} = \underbrace{F}_{\substack{\text{evaluated}\\\text{numerically}}}(\eta)\,; \quad \eta = y\sqrt{\frac{U_o}{\nu x}}; \quad y \equiv \eta\sqrt{\frac{\nu x}{U_o}}; \quad \frac{y}{x} = \frac{\eta}{\sqrt{\underbrace{R_x}_{\text{local R\#}}}}$$

$$\left.\begin{aligned}
&\delta \equiv \sqrt{\frac{\nu x}{U_o}} \\
&\delta_{.99} \cong 4.9\sqrt{\frac{\nu x}{U_o}}, \text{ i.e., } \eta_{.99} = 4.9 \\
&\delta^* \cong 1.72\sqrt{\frac{\nu x}{U_o}}, \text{ i.e., } \eta^* = 1.72 \\
&\theta \cong 0.664\sqrt{\frac{\nu x}{U_o}}
\end{aligned}\right\}
\begin{aligned}
&\delta \propto \sqrt{x},\ \delta \propto 1\Big/\sqrt{U_o} \\
&\frac{\delta}{x} \propto \sqrt{\frac{\nu}{U_o x}} = \frac{1}{\sqrt{R_x}}
\end{aligned}$$

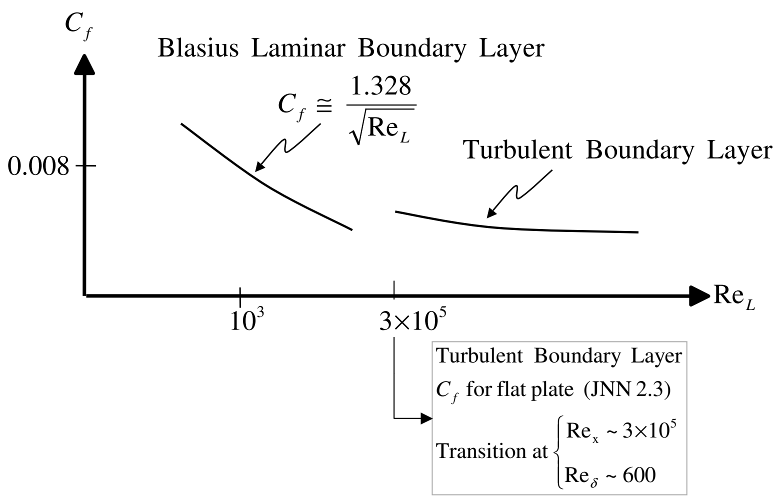
$$\left.\begin{aligned}
\tau_o \equiv \tau_w &\cong 0.332\rho U_o^2\left(\frac{U_o x}{\nu}\right)^{-1/2} \\
&= 0.332\left(\rho U_o^2\right)\underbrace{R_x^{-1/2}}_{\text{local R\#}}
\end{aligned}\right\}
\begin{aligned}
&\tau_o \propto \frac{1}{\sqrt{x}} \\
&\tau_o \propto U_o^{3/2}
\end{aligned}$$

Total drag on plate L x B

$$D = \underbrace{B}_{\text{width}} \int_0^L \tau_o dx \cong 0.664 \left(\rho U_o^2\right) (BL) \underbrace{\left(\frac{U_o L}{\nu}\right)^{-1/2}}_{R_{e_L}^{-1/2}} \Rightarrow \; D \propto \sqrt{L}, \quad D \propto U^{3/2}$$

Friction (drag) coefficient:

$$C_f = \frac{D}{\frac{1}{2}\left(\rho U_o^2\right)(BL)} \cong \frac{1.328}{\sqrt{R_{e_L}}} \; \Rightarrow \; C_f \propto \frac{1}{\sqrt{L}}, \quad C_f \propto \frac{1}{\sqrt{U}}$$



Skin friction coefficient as a function of R_e.

A look ahead: Turbulent Boundary Layers

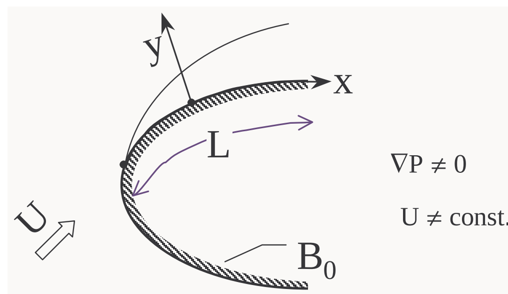
Observe form the previous figure that the function $C_{f,\ laminar}(R_e)$ for a laminar boundary layer is different from the function $C_{f,\ turbulent}(R_e)$ for a turbulent boundary layer for flow over a flat plate.

Turbulent boundary layers will be discussed in proceeding Lecture.

4.8 Laminar Boundary Layers for Flow Over a Body of General Geometry

The velocity profile given in BLBL is the **exact** velocity profile for a steady, laminar flow over a flat plate. What is the velocity profile for a flow over any arbitrary body? In general it is $dp/dx \neq 0$ and the boundary layer governing equations cannot be easily solved as was the case for the BLBL. In this paragraph we will describe a typical *approximative* procedure used to solve the problem of flow over a body of general geometry.

1. Solve P-Flow outside $B \equiv B_0$
2. Solve boundary layer equations (with ∇P term) $\rightarrow$ get $\delta^*(x)$
3. From $B_0 + \delta^* \rightarrow B$
4. Repeat steps (1) to (3) until no change



- **von Karman's zeroth moment integral equation**

$$\boxed{\frac{\tau_0}{\rho} = \frac{d}{dx}\left(U^2(x)\theta(x)\right) + \delta^*(x)U(x)\frac{dU}{dx}} \qquad (4)$$

- **Approximate solution method due to Polthausen for general geometry** ($dp/dx \neq 0$) **using von Karman's momentum integrals.**

  The basic idea is the following: we *assume* an *approximate* velocity profile (e.g. linear, 4^{th} order polynomial, ...) in terms of an *unknown* parameter $\delta(x)$. From the velocity profile we can immediately calculate δ^*, θ and τ_o as functions of $\delta(x)$ and the P-Flow velocity $U(x)$.

  Independently from the boundary layer approximation, we obtain the P-Flow solution outside the boundary layer $U(x)$, $\frac{dU}{dx}$.

  Upon substitution of $\delta^*, \theta, \tau_o, U(x)$, $\frac{dU}{dx}$ in von Karman's moment integral equation(s) we form an ODE for δ in terms of x.

- **Example for a 4^{th} order polynomial Polthausen velocity profile**

  Polthausen profiles - a family of profiles as a function of a single parameter $\Lambda(x)$ (shape function factor).

  - ☐ Assume an approximate velocity profile, say a 4^{th} order polynomial:

    $$\frac{u\left(x,y\right)}{U\left(x,0\right)} = a\left(x\right)\frac{y}{\delta} + b\left(x\right)\left(\frac{y}{\delta}\right)^2 + c\left(x\right)\left(\frac{y}{\delta}\right)^3 + d\left(x\right)\left(\frac{y}{\delta}\right)^4 \qquad (5)$$

    There can be no constant term in (5) for the no-slip BC to be satisfied $y = 0$, i.e, $u(x,0) = 0$.
    We use three BC's at $y = \delta$

    $$\frac{u}{U} = 1, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{at } y = \delta \qquad (6)$$

    From (6) in (5), we re-write the coefficients $a(x)$, $b(x)$, $c(x)$ and $d(x)$ in terms of $\Lambda(x)$

    $$a = 2 + \Lambda/6, \ b = -\Lambda/2, \ c = -2 + \Lambda/2, \ d = 1 - \Lambda/6$$

  - ☐ To specify the *approximate* velocity profile $\frac{u(x,y)}{U(x,0)}$ in terms of a single unknown parameter δ we use the x-momentum equation at $y = 0$, where $u = v = 0$

    $$\underset{\downarrow \atop 0}{u}\frac{\partial u}{\partial x} + \underset{\downarrow \atop 0}{v}\frac{\partial u}{\partial y} = \underbrace{U\frac{\partial U}{\partial x}}_{-\frac{1}{\rho}\frac{dp}{dx}} + \underbrace{\nu\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}}_{\nu\frac{2bU}{\delta^2}} \Rightarrow b = -\frac{1}{2}\left(\frac{dU}{dx}\frac{\delta^2}{\nu}\right) \Rightarrow \boxed{\Lambda(x) = \frac{dU}{dx}\frac{\delta^2(x)}{\nu}}$$

    Observe: $\Lambda \propto \frac{dU}{dx} \Rightarrow \begin{cases} \Lambda > 0: & \text{favorable pressure gradient} \\ \Lambda < 0: & \text{adverse pressure gradient} \end{cases}$

    Putting everything together:

    $$\begin{aligned} \frac{u\left(x,y\right)}{U\left(x,0\right)} &= 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4 + \\ &+ \frac{dU}{dx}\frac{\delta^2}{\nu}\left[\frac{1}{6}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^2 + \frac{1}{2}\left(\frac{y}{\delta}\right)^3 - \frac{1}{6}\left(\frac{y}{\delta}\right)^4\right] \end{aligned}$$

☐ Once the *approximate* velocity profile $\frac{u(x,y)}{U(x,0)}$ is given in terms of a single unknown parameter $\delta(x)$, then δ^*, θ and τ_o are evaluated

$$\begin{aligned}
\delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \delta\left(\frac{3}{10} - \frac{1}{120}\left(\frac{dU}{dx}\frac{\delta^2}{\nu}\right)\right) \\
\theta &= \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \delta\left(\frac{37}{315} - \frac{1}{945}\left(\frac{dU}{dx}\frac{\delta^2}{\nu}\right) - \frac{1}{9072}\left(\frac{dU}{dx}\frac{\delta^2}{\nu}\right)^2\right) \\
\tau_o &= \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\mu U}{\delta}\left(2 + \frac{1}{6}\left(\frac{dU}{dx}\frac{\delta^2}{\nu}\right)\right)
\end{aligned}$$

Notes:

- Incipient flow ($\tau_o = 0$) for $\Lambda = -12$. *However, recall that once the flow is separated the boundary layer theory is no longer valid.*

- For $\frac{dU}{dx} = 0 \rightarrow \Lambda = 0$ Pohlhausen profile differs from Blasius LBL only by a few percent.

☐ After we solve the P-Flow and determine $U(x)$, $\frac{dU}{dx}$ we substitute everything into von Karman's momentum integral equation (4) to obtain

$$\frac{d\delta}{dx} = \frac{1}{U}\frac{dU}{dx}g(\delta) + \frac{d^2U/dx^2}{dU/dx}h(\delta)$$

where g, h are **known** rational polynomial functions of δ.
This is an ODE for $\delta = \delta(x)$ where $U, \frac{dU}{dx}, \frac{d^2U}{dx^2}$ are specified from the P-Flow solution.
General procedure:

1. Make a reasonable approximation in the form of (5),
2. Apply sufficient BC's at $y = \delta$, and the x-momentum at $y = 0$ to reduce (5) as a function a **single** unknown δ,
3. Determine $U(x)$ from P-Flow, and
4. Finally substitute into Von Karman's equation to form an ODE for $\delta(x)$. Solve either analytically or numerically to determine the boundary layer growth as a function of x.