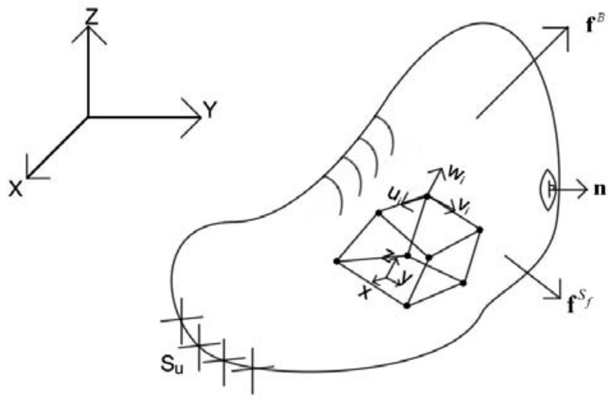
2.092/2.093 — Finite Element Analysis of Solids & Fluids I Fall '09

Lecture 5 - The Finite Element Formulation

Prof. K. J. Bathe MIT OpenCourseWare

In this system, (X, Y, Z) is the global coordinate system, and (x, y, z) is the local coordinate system for the element i.



We want to satisfy the following equations:

$$\left.\begin{array}{ll}\tau_{ij,j} + f_i^B = 0 & \text{in } V \\ \tau_{ij} n_j = f_i^{S_f} & \text{on } S_f\end{array}\right\} \rightarrow \text{Equilibrium Conditions}$$

$$u_i \big|_{S_u} = u_i^{S_u} \quad \rightarrow \quad \text{Compatibility Conditions} \tag{A}$$

$$\tau_{ij} = f(\varepsilon_{kl}) \quad \rightarrow \quad \text{Stress-strain Relations}$$

Then we have the exact solution.

Principle of Virtual Displacements

$$\int_V \overline{\boldsymbol{\varepsilon}}^T \boldsymbol{C} \boldsymbol{\varepsilon} dV = \int_V \overline{\boldsymbol{u}}^T \boldsymbol{f}^{\boldsymbol{B}} dV + \int_{S_f} \overline{\boldsymbol{u}}^{S_f T} \boldsymbol{f}^{S_f} dS_f \tag{B}$$

Here, real stresses ($\boldsymbol{C}\boldsymbol{\varepsilon}$) are in equilibrium with the external forces ($\boldsymbol{f}^B$, $\boldsymbol{f}^{S_f}$). Note that Eq. (B) is equivalent to Eq. (A). Recall that we defined

$$\boldsymbol{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}$$

$$\overline{\boldsymbol{\varepsilon}}^T = \begin{bmatrix} \bar\varepsilon_{xx} & \bar\varepsilon_{yy} & \bar\varepsilon_{zz} & \bar\gamma_{xy} & \bar\gamma_{yz} & \bar\gamma_{zx} \end{bmatrix} = \left[\frac{\partial \overline{u}}{\partial x} \cdots \right]$$

Basic assumptions:

$$\boldsymbol{u}^{(m)} = \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{bmatrix}^{(m)} = \underset{3\times n}{\boldsymbol{H}}^{(m)} \underset{n\times 1}{\hat{\boldsymbol{u}}} \tag{1}$$

$$\hat{\boldsymbol{u}} = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_N \\ v_N \\ w_N \end{bmatrix}$$

N is the number of nodes ($3N = n$) and $\boldsymbol{H}$ is the displacement interpolation matrix. For the moment, let's assume $S_u = 0$. We use

$$\hat{\boldsymbol{u}}^T = \begin{bmatrix} u_1 & u_2 & u_3 & \dots & u_n \end{bmatrix}$$

Then, we obtain

$$\underset{6\times 1}{\boldsymbol{\varepsilon}}^{(m)} = \underset{6\times n}{\boldsymbol{B}}^{(m)} \underset{n\times 1}{\hat{\boldsymbol{u}}} \tag{2}$$

We also assume

$$\bar{\boldsymbol{u}}^{(m)} = \boldsymbol{H}^{(m)}\bar{\hat{\boldsymbol{u}}} \tag{3}$$

$$\underset{6\times 1}{\bar{\boldsymbol{\varepsilon}}}^{(m)} = \underset{6\times n}{\boldsymbol{B}}^{(m)} \underset{n\times 1}{\bar{\hat{\boldsymbol{u}}}} \tag{4}$$

where $\boldsymbol{B}$ is the strain-displacement matrix. Substitute equations (1) through (4) into (B):

$$\sum_m \int_V \bar{\boldsymbol{\varepsilon}}^{(m)T} \boldsymbol{C}^{(m)} \boldsymbol{\varepsilon}^{(m)} dV^{(m)} = \sum_m \int_V \bar{\boldsymbol{u}}^{(m)T} \boldsymbol{f}^{B(m)} dV^{(m)} + \sum_m \sum_i \int_{S_f^{i(m)}} \bar{\boldsymbol{u}}^{S_f^{i(m)}T} \boldsymbol{f}^{S_f^{i(m)}} dS_f^{i(m)} \tag{B*}$$

where i sums over the element surfaces composing $S_f^{(m)}$. The equation now becomes

$$\bar{\hat{\boldsymbol{u}}}^T \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} dV^{(m)} \right\} \hat{\boldsymbol{u}} = \bar{\hat{\boldsymbol{u}}}^T \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{H}^{(m)T} \boldsymbol{f}^{B(m)} dV^{(m)} + \sum_m \sum_i \int_{S_f^{i(m)}} \boldsymbol{H}^{S_f^{i(m)}T} \boldsymbol{f}^{S_f^{i(m)}} dS_f^{i(m)} \right\}$$

$\hat{\boldsymbol{u}}$ is the unknown to be found. When evaluated on $S_f^{i(m)}$,

$$\bar{\boldsymbol{u}}^{S_f^{i(m)}} = \boldsymbol{H}^{S_f^{i(m)}} \bar{\hat{\boldsymbol{u}}}$$

$$\boldsymbol{H}^{S_f^{i(m)}} = \boldsymbol{H}^{(m)} \big|_{S_f^{i(m)}}$$

With the transformed equation above, we can insert the following identity matrices:

Let $\bar{\hat{\boldsymbol{u}}}^T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \end{bmatrix}$ $\rightarrow$ Gives the first equation to solve for
Then $\bar{\hat{\boldsymbol{u}}}^T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix}$ $\rightarrow$ Gives the second equation
Then $\bar{\hat{\boldsymbol{u}}}^T = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \end{bmatrix}$ $\rightarrow$ Gives the third equation
... and so on.

We finally obtain $\boldsymbol{K}\hat{\boldsymbol{u}} = \boldsymbol{R}$. Now, let's drop off the hat!

$$\boxed{\boldsymbol{K}\boldsymbol{U}=\boldsymbol{R}}$$

$$\boldsymbol{K}=\sum_m \boldsymbol{K}^{(m)} \quad ; \quad \boldsymbol{K}^{(m)}=\int_{V^{(m)}} \boldsymbol{B}^{(m)T}\boldsymbol{C}^{(m)}\boldsymbol{B}^{(m)}dV^{(m)}$$

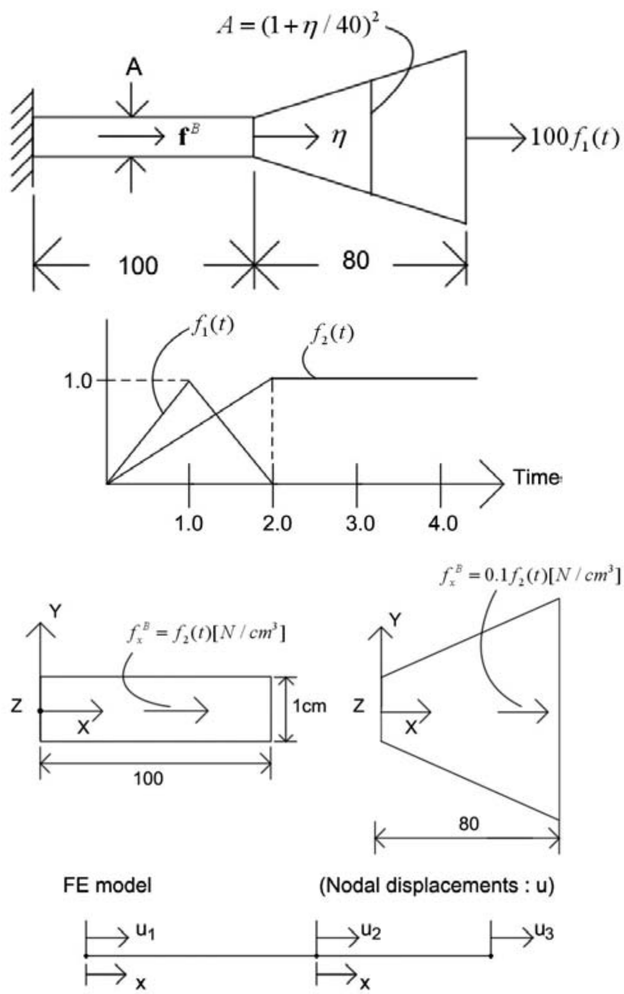
$$\boldsymbol{R}=\boldsymbol{R}_B+\boldsymbol{R}_S$$

$$\boldsymbol{R}_B=\sum_m \boldsymbol{R}_B^{(m)} \quad ; \quad \boldsymbol{R}_B^{(m)}=\int_{V^{(m)}} \boldsymbol{H}^{(m)T}\boldsymbol{f}^{B(m)}dV^{(m)}$$

$$\boldsymbol{R}_S=\sum_m \boldsymbol{R}_S^{(m)} \quad ; \quad \boldsymbol{R}_S^{(m)}=\sum_i \int_{S_f^{i(m)}} \boldsymbol{H}^{S_f^{i(m)}T}\boldsymbol{f}^{S_f^{i(m)}}dS_f^{i(m)}$$

Example 4.5

Reading assignment: Section 4.2



For this system, we can define $\boldsymbol{U}^T=[\,u_1 \quad u_2 \quad u_3\,]$. We want to find:

$$\boldsymbol{u}^{(1)}(x)=\boldsymbol{H}^{(1)}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \quad ; \quad \boldsymbol{u}^{(2)}(x)=\boldsymbol{H}^{(2)}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
Fall 2009