Homework #2 Solution

4.2 (a)

$$\int_V \overline{\underline{\epsilon}}^T \underline{\tau}\, dV = \int_0^L \left(\frac{d\bar{u}}{dx}\right) \tau A(x)\, dx \quad \text{where} \quad \tau = E\frac{du}{dx}$$

$$\int_V \underline{\bar{u}}^T \underline{f}^B dV = 0\,, \quad \int_S \underline{\bar{U}}^{S^T} \underline{f}^S ds = \bar{U}\big|_{x=L} F$$

$\therefore$ Principle of virtual displacements gives

$$\int_0^L \left(\frac{d\bar{u}}{dx}\right) \tau A(x)\, dx = \bar{u}_L F \qquad \text{———} \quad (*)$$

(b) (i) $\bar{U}(x) = a_0 x \qquad \therefore\ \bar{\epsilon}(x) = a_0$

From equation $(*)$,

$$(L.H.S) = F \cdot \int_0^L a_0 \left(\frac{72}{73} + \frac{24x}{73L}\right)\left(1 - \frac{x}{4L}\right) dx = a_0 F L$$

$$(R.H.S) = a_0 F L \qquad \text{'OK'}$$

(ii) $\bar{U}(x) = a_0 x^2 \qquad \therefore\ \bar{\epsilon}(x) = 2a_0 x$

Similarly, $(L.H.S) = 2F\int_0^L a_0 x \left(\frac{72}{73} + \frac{24x}{73L}\right)\left(1 - \frac{x}{4L}\right) dx = a_0 F L^2$

$$(R.H.S) = a_0 F L^2 \qquad \text{'OK'}$$

(iii) $\bar{U}(x) = a_0 x^3\,, \quad \bar{\epsilon}(x) = 3a_0 x^2$

Similarly, $(L.H.S) = \dfrac{729}{730} a_0 F L^2$

$(R.H.S) = a_0 F L^2 \quad \therefore\ (L.H.S) \neq (R.H.S)$ in this case

Hence the given τ is not the exact solution of the mathematical model !

(c) $E \frac{\partial}{\partial x}\left(A \frac{\partial u}{\partial x}\right) = 0$ with $EA \frac{\partial u}{\partial x}\Big|_{x=L} = F$

$\rightarrow A \frac{\partial u}{\partial x} = C$ (constant) $C = \frac{F}{E}$ (from b.c)

$\therefore \frac{\partial u}{\partial x} = \frac{F}{EA_0}\left[1 / \left(1 - \frac{x}{4L}\right)\right]$

$\therefore \tau = E \frac{\partial u}{\partial x} = \frac{F}{A_0}\left[1 / \left(1 - \frac{x}{4L}\right)\right]$

(d) Every given $\bar{U}(x)$ satisfies the essential b.c.

For any $\bar{\epsilon} = \frac{\partial \bar{U}(x)}{\partial x}$

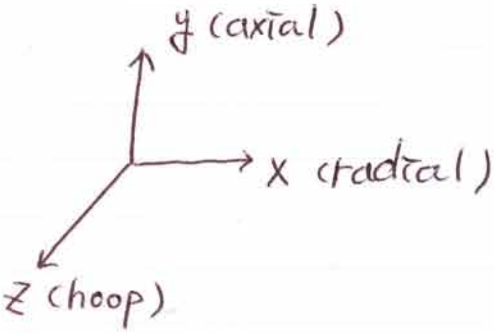
$$\int_0^L \bar{\epsilon}\left[\frac{F}{A_0} / \left(1 - \frac{x}{4L}\right)\right]\left[A_0\left(1 - \frac{x}{4L}\right)\right] dx = \int_0^L \bar{\epsilon} F\, dx = \bar{U}(L) \cdot F$$

That is, the equation (*) in part (a) holds.

4.6 In this problem. ε_{xx} and ε_{zz} are only considered. Hence, the matrix $\underline{C}$ is given by

$$\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

And $$\underline{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ u/x \end{bmatrix}$$



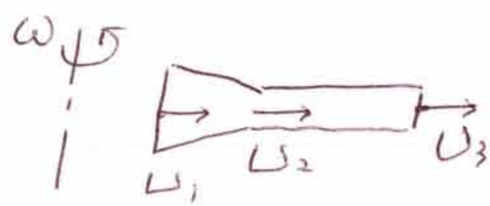
For each element

$$\underline{H}^{(1)} = \begin{bmatrix} 1-\frac{\eta}{60} & \frac{\eta}{60} & 0 \end{bmatrix}, \quad \underline{H}^{(2)} = \begin{bmatrix} 0 & 1-\frac{\eta}{80} & \frac{\eta}{80} \end{bmatrix}$$

$$\underline{B}^{(1)} = \begin{bmatrix} -\frac{1}{60} & \frac{1}{60} & 0 \\ \frac{1-\frac{\eta}{60}}{(\eta+20)} & \frac{\frac{\eta}{60}}{(\eta+20)} & 0 \end{bmatrix}, \quad \underline{B}^{(2)} = \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \\ 0 & \frac{1-\frac{\eta}{80}}{\eta+80} & \frac{\frac{\eta}{80}}{\eta+80} \end{bmatrix}$$

$$t^{(1)} = 3\left(1-\frac{\eta}{90}\right), \quad t^{(2)} = 1 \qquad (t = \text{thickness})$$

Here, $\underline{U}^T = [U_1 \; U_2 \; U_3]$



Then, $\underline{K} = \int_0^{60} \underline{B}^{(1)T} \underline{C}\, \underline{B}^{(1)} \cdot 1 \cdot 3 \left(1 - \frac{\eta}{90}\right)(\eta + 20)\, d\eta$

$$+ \int_0^{80} \underline{B}^{(2)T} \underline{C}\, \underline{B}^{(2)} \cdot 1 \cdot 1 \cdot (\eta + 80)\, d\eta$$

$$\underline{R} = \underline{R}_B = \int_0^{60} \underline{H}^{(1)T} \cdot \rho\omega^2 (\eta + 20) \cdot 1 \cdot 3 \left(1 - \frac{\eta}{90}\right)(\eta + 20)\, d\eta$$

$$+ \int_0^{80} \underline{H}^{(2)T} \cdot \rho\omega^2 (\eta + 80) \cdot 1 \cdot 1 \cdot (\eta + 80)\, d\eta$$

Note that the matrices are obtained for a unit radian.