Session 17

For systems w/ ideal constraints $\int_{t_1}^{t_2} (\delta T + \delta W)\big|_{\underline{r}(t)} dt = 0$

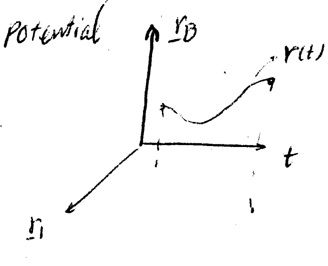
if in addition, all forces are potential

$\boxed{\delta I = 0}$

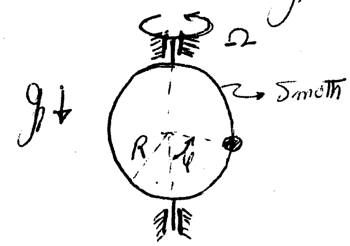
action

$I = \int_{t_1}^{t_2} L(\underline{r}(t), \dot{\underline{r}}(t)) dt$

$L = T - V$

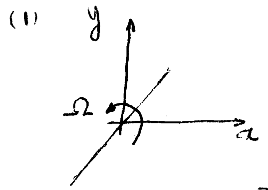


Example: Bead moving on rotating ring



- Active forces: gravity (potential)
- Constraints:

(1)



$\frac{y}{x} = \tan \Omega t$

i.e. $y \cos \Omega t - x \sin \Omega t = 0$

(2) $x^2 + y^2 + z^2 - R^2 = 0$

2 holonomic constraints
$\Rightarrow$ System is holonomic

Principle of least action applies $\delta I = 0$

\# DOF = 3 - 2 = 1

$L = T - V = \frac{1}{2} m |\dot{\underline{r}}|^2 + (+mgR \cos \varphi)$

$= \frac{1}{2} m ((R \sin \varphi \, \Omega)^2 + R^2 \dot{\varphi}^2) + mgR \cos \varphi$

$\delta I = \delta \int_{t_1}^{t_2} [\frac{1}{2} m (R^2 \sin^2 \varphi \, \Omega^2 + R^2 \dot{\varphi}^2) + mgR \cos \varphi] dt$

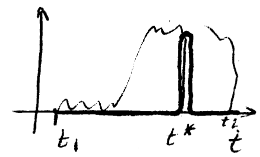
$= \int_{t_1}^{t_2} [\frac{1}{2} m (R^2 \Omega^2 2 \sin \varphi \cos \varphi \, \delta \varphi + R^2 2 \dot{\varphi} \delta \dot{\varphi}) - mgR \sin \varphi \, \delta \varphi] dt$

NOTE: $\int_{t_1}^{t_2} (\overset{u}{\dot{\varphi}} \overset{v'}{\delta \dot{\varphi}}) dt = [\dot{\varphi} \delta \varphi]_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\varphi} \, \delta \varphi \, dt$

$\delta I = [\frac{1}{2} m (R^2 \Omega^2 \sin 2\varphi \pm mR^2 \ddot{\varphi}) - mgR \sin \varphi] \delta \varphi \, dt = 0$

\- must vanish for all t_1 and t_2 - for kinematically admissible $\delta \varphi$

[ ]



is $[\quad] = 0$ for all t?
assume not $\Rightarrow [\quad]|_{t=t^*} \neq 0$ for some t^*

$\Rightarrow \delta I \neq 0$ ↯

$\Rightarrow mR^2\ddot{\varphi} + mgR\sin\varphi - \frac{1}{2}mR^2\sin 2\varphi = 0$

NOTE: Constraint force did not enter calculation

the above calculation can be performed for general systems with ideal holonomic constraints
Extended Hamilton's Principle

$$\int_{t_1}^{t_2} (\delta T + \delta W)\Big|_{x(t)} dt = \int_{t_1}^{t_2} \Big[\delta T + \underbrace{(-\delta V)}_{\text{potential force}} + \underbrace{\sum_{j=1}^{N} Q_j \delta q_j}_{\text{non-potential}} + \delta W^{Const}\Big] dt$$

$$= \int_{t_1}^{t_2} \Big(\delta L + \sum_{j=1}^{N} Q_j \delta q_j\Big) dt = \sum_{j=1}^{N} \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j + Q_j \delta q_j\Big] dt$$

$L(q, \dot{q}; t)$

NOTE: $\displaystyle\int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial \dot{q}} \delta \dot{q}_j\Big) dt = \underbrace{\Big[\frac{\partial L}{\partial \dot{q}_j}\delta q_j\Big]_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\delta q_j \, dt$

$$\sum_{j=1}^{N} \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + Q_j\Big]\delta q_j \, dt = 0$$

This must hold for all t_1, t_2
for all δq_j, $j = 1, \dots, N$

For any j, repeat argument from the previous example, setting
$\delta q_1, \delta q_2, \dots, \delta q_{j-1}, \delta q_{j+1}, \dots, \delta q_N = 0$

Here we use heavily the independence of q_j's (system is holonomic)

$$\Rightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j} \quad j = 1, \dots, N \qquad N: \text{\#DOF}$$

$L = T - V$