Massachusetts Institute of Technology

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6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 4 Solutions[1]

Problem 4.1

FIND A FUNCTION $V : \mathbf{R}^3 \mapsto \mathbf{R}_+$ WHICH HAS A UNIQUE MINIMUM AT $\bar{x} = 0$, AND IS STRICTLY MONOTONICALLY DECREASING ALONG ALL NON-EQUILIBRIUM TRAJECTORIES OF SYSTEM

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + x_2(t)^2,\\ \dot{x}_2(t) &= -x_2(t)^3 + x_3(t)^4,\\ \dot{x}_3(t) &= -x_3(t)^5.\end{aligned}$$

Let us begin with collecting storage function and compatible quadratic supply rate pairs for the system. Naturally, positive definite functions of system states are a good starting point. For $V_1(x) = x_1^2$ we have

$$\dot{V}_1 = -2x_1^2 + 2x_1x_2^2 \le -x_1^2 + w_1^2 = \sigma_1,$$

where $w_1 = x_2^2$, and the classical inequality

$$2ab \le a^2 + b^2$$

was used. For $V_2 = x_2^2$ we have

$$\dot{V}_2 = -2x_2^4 + 2x_2x_3^4 \le -w_1^2 + 2w_2^2 = \sigma_2,$$

where $w_2 = x_3^{8/3}$ and the inequality

$$2ab^3 \le a^4 + 2b^4$$

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(a weakened version of a classical inequality) was used. Finally, for $V_3 = 3x_3^{4/3}$ we have

$$\dot{V}_3 = -4x_3^{16/3} = -4w_2^2.$$

Now, for

$$V = c_1V_1 + c_2V_2 + c_3V_3$$

we have

$$\dot{V} \le c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3 = -c_1x_1^2 + (c_1 - c_2)w_1^2 + (2c_2 - 4c_3)w_2^2.$$

Taking $c_1 = 1$, $c_2 = c_3 = 2$ yields a continuously differentiable Lyapunov function

$$V(x) = x_1^2 + 2x_2^2 + 6x_3^{4/3}$$

for which the derivatives along system trajectories are bounded by

$$\dot{V}(x) \le -x_1^2 - x_2^4 - 4x_3^{16/3}.$$

Problem 4.2

System Δ takes arbitrary continuous input signals $v: [0,\infty) \mapsto \mathbf{R}$ and produces continuous outputs $w: [0,\infty) \mapsto \mathbf{R}$ in such a way that the series connection of Δ and the LTI system with transfer function $G_0(s) = 1/(s+1)$, described by equations

$$\dot{x}_0(t) = -x_0(t) + w(t), \quad w(\cdot) = \Delta(v(\cdot)),$$

has a non-negative storage function with supply rate

$$\sigma_0(\bar{x}_0, \bar{v}, \bar{w}) = (\bar{w} - 0.9\bar{x}_0)(\bar{v} - \bar{w}).$$

(a) Find at least one *nonlinear* system Δ which fits the description.

The ideal saturation nonlinearity

$$\mathrm{sat}(y) = \begin{cases} y/|y|, & |y| \ge 1, \\ y, & |y| \le 1, \end{cases}$$

is a nice example of Δ satisfying the conditions. Indeed, if

$$\dot{x}_0(t) = -x_0(t) + \mathrm{sat}(v(t)), \quad x_0(0) = 0$$

then $|x_0(t)| \le 1$ for all $t \ge 0$. Hence

$$(v(t) - \mathrm{sat}(v(t)))(\mathrm{sat}(v(t)) - x_0(t)) \ge 0 \quad \forall\, t \ge 0$$

(if $v(t) \in [-1, 1]$ then the product equals zero, otherwise the multipliers have same sign).

(b) DERIVE CONSTRAINTS TO BE IMPOSED ON THE VALUES $G(j\omega)$ OF A TRANSFER FUNCTION

$$G(s) = C(sI - A)^{-1}B$$

WITH A HURWITZ MATRIX A, WHICH GUARANTEE THAT $x(t) \to 0$ AS $t \to \infty$ FOR EVERY SOLUTION OF

$$\dot{x}(t) = Ax(t) + Bw(t), \;\; v(t) = Cx(t), \;\; w(\cdot) = \Delta(v(\cdot)).$$

MAKE SURE THAT YOUR CONDITIONS ARE SATISFIED AT LEAST FOR ONE NON-ZERO TRANSFER FUNCTION $G = G(s)$.

Let us prove that condition

$$\mathrm{Re}\left[(1 - G(j\omega))\frac{0.1 - j\omega}{1 - j\omega}\right] > 0 \;\; \forall\, \omega \in \mathbf{R} \tag{4.1}$$

is sufficient to guarantee that $x(t) \to 0$ as $t \to \infty$. Indeed, since A is a Hurwitz matrix, and G is strictly proper, there exists $\epsilon > 0$ such that

$$\mathrm{Re}\left[(1 - G(j\omega))\frac{0.1 - j\omega}{1 - j\omega}\right] > \epsilon(1 + |(j\omega I - A)^{-1}B|^2) \;\; \forall\, \omega \in \mathbf{R}.$$

Therefore, the frequency inequality conditions of the KYP Lemma are satisfied for the existence of a matrix $P = P'$ such that

$$2\begin{bmatrix} x \\ x_0 \end{bmatrix}' P \begin{bmatrix} Ax + Bw \\ w - x_0 \end{bmatrix} \leq (w - Cx)(w - 0.9x_0) - \epsilon(|x|^2 + |w|^2) \;\; \forall\, w, x_0 \in \mathbf{R}, \; x \in \mathbf{R}^n.$$

To show that P is positive definite, substitute $w = x_0$ into the last inequality, which yields

$$2\begin{bmatrix} x \\ x_0 \end{bmatrix}' P \begin{bmatrix} Ax + Bx_0 \\ -0.1x_0 \end{bmatrix} \leq -\epsilon(|x|^2 + |0.9x_0|^2) \;\; \forall\, x_0 \in \mathbf{R}, \; x \in \mathbf{R}^n,$$

which is equivalent to the Lyapunov inequality

$$P\hat{A} + \hat{A}'P = -Q,$$

where

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & -0.1 \end{bmatrix}, \quad Q = \epsilon \begin{bmatrix} I & 0 \\ 0 & 0.81 \end{bmatrix}.$$

Since $\hat{A}$ is a Hurwitz matrix, and $Q = Q' > 0$, it follows that $P > 0$.

Now

$$V = V_0 + \begin{bmatrix} x \\ x_0 \end{bmatrix}' P \begin{bmatrix} x \\ x_0 \end{bmatrix}$$

is a non-negative storage function for the closed loop system, with supply rate

$$\sigma = -\epsilon(|x|^2 + |w|^2).$$

Hence w is square integrable over the interval $[0, \infty)$. Since

$$\dot{x} = Ax + Bw,$$

and A is a Hurwitz matrix, this implies that $x(t) \to 0$ as $t \to \infty$.

Since

$$\text{Re}\frac{0.1 - j\omega}{1 - j\omega} \geq 0.1 \quad \forall\, \omega \in \mathbf{R},$$

condition (4.1) is satisfied for all G with sufficiently small H-Infinity norm (maximal absolute value of the frequency response).

Problem 4.3

For the pendulum equation

$$\ddot{y}(t) + \dot{y} + sin(y) = 0,$$

find a single continuously differentiable Lyapunov function $V = V(y, \dot{y})$ that yields the maximal region of attraction of the equilibrium $y = \dot{y} = 0$. (In other words, the level set

$$\{\bar{x} \in \mathbf{R}^2 : \ V(\bar{x}) < 1\}$$

schould be a union of disjoint open sets, one of which is the attractor Ω of the zero equilibrium, and $V(y(t), \dot{y}(t))$ schould have negative derivative at all points of Ω except the origin.)

Note that the problem can be interpreted as follows: given the initial angular position and angular velocity of a pendulum, find the number of complete rotations it will have before settling at an equilibrium position. An "exact analytical" answer can be obtained by stating that the maximal region of attraction is the area bounded by the four separatrix solutions of the system equation, converging to the two unstable equilibria $(0, \pm\pi)$. However, this "exact" answer (which cannot be expressed in elementary functions) will be of no use in the case when the pendulum model is slightly modified (a different friction model, flexibility of the pendulum taken into account, etc.) On the other hand, one can expect that an estimate obtained by using a Lyapunov function will be more "robust" with respect to various perturbations of the model.

An obvious Lyapunov function is given by the system energy (potential plus kinetic)

$$V_0(y, \dot{y}) = 0.5\dot{y}^2 - \cos(y), \ dV/dt = -\dot{y}(t)^2.$$

To estimate the region of attraction of the equilibrium at the origin, using this Lyapunov function, one may can find a constant c such that the level set

$$\mathcal{L}(V_0, c) = \{(y_0, y_1) : \ V_0(y_0, y_1) < c\}$$

does not contain a path connecting the origin with any other equilibrium of the system. It is easy to see that taking $c = 1$ does the job, and yields the region of attraction Ω_0 given by

$$\Omega_0 = \{(y, \dot{y}) : \ 0.5\dot{y}^2 - \cos(y) < 1, \ -\pi < y < \pi\}.$$

This appears to be a very poor estimate, taking into account what we know about the true maximal region of attraction.

To get a better Lyapunov function, one can try to construct it in such a way that the level sets are polytopes centered at the origin. Remember that a function V is a Lyapunov function if and only if the system trajectories never leave any of its level sets. Since the boundary of a polytope in $\mathbf{R}^2$ is a segment, it is especially easy to check this condition for the Lyapunov functions candidates with polytopic level sets.

One of the simplest examples of a Lyapunov function constructed this way is given by

$$V_1(y, \dot{y}) = \begin{cases} |y| + |\dot{y}|, & y\dot{y} \geq 0, \\ \max\{y, \dot{y}\}, & y\dot{y} \leq 0. \end{cases}$$

It is easy to check that V_1 is a Lyapunov function for the pendulum system in the area

$$\Omega_1 = \{(y, \dot{y}) : \ V_1(y, \dot{y}) < \pi\},$$

which is also the resulting estimate of the region of attraction.

The previous estimate Ω_0 is contained in Ω_1. Even better estimates can be obtained by using other lyapunov functions with polytopic level sets.