2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
Spring Term 2008

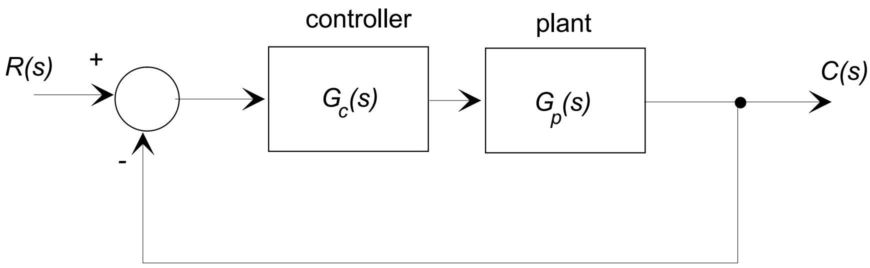
Lecture 24[1]

Reading:

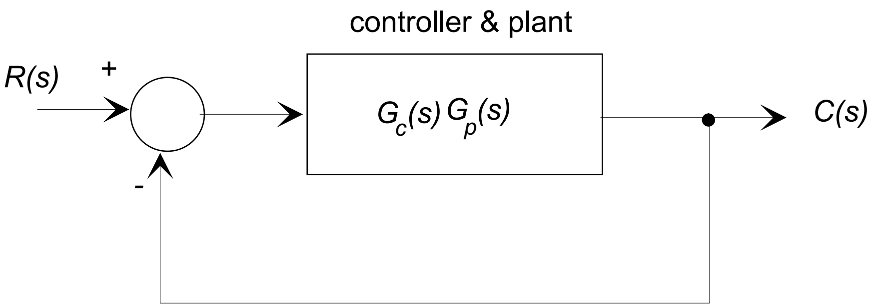
- Nise: Chapter 7

1 The Poles and Zeros of Closed-loop systems:

Consider the unity feedback system shown below with a controller $G_c(s)$ and plant $G_p(s)$:



Combine the two cascaded blocks to form a single forward transfer function $G_f(s) = G_c(s)G_p(s)$



and write

$$G_f(s) = \frac{N_f(s)}{D_f(s)}$$

in terms of the numerator polynomial $N_f(s)$ and denominator polynomial $D_f(s)$. The closed-loop transfer function is

$$G_{cl}(s) = \frac{G_f(s)}{1 + G_f(s)} = \frac{N_f(s)}{D_f(s) + N_f(s)}$$

from which we see that

- The closed-loop poles are the roots of the characteristic equation $N_f(s) + D_f(s) = 0$.
- The closed-loop zeros are the same as the zeros of the forward transfer function.

■ Example 1

Find the closed-loop transfer function of the plant $G_p(s) = 3/(s+3)$ under P-D control where $G_c = 10 + 2s$.

The forward transfer function is

$$G_f(s) = G_c(s)G_p = \frac{6(5+s)}{s+3}$$

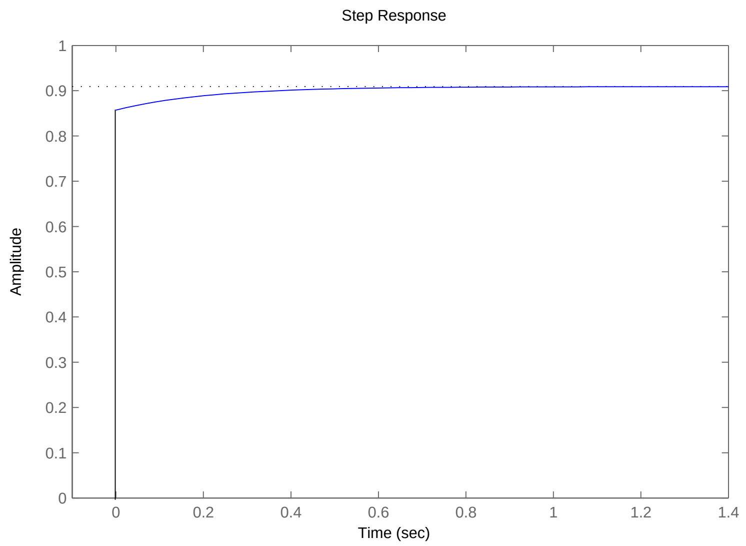
The closed-loop transfer function is:

$$G_{cl}(s) = \frac{N_f(s)}{D_f(s) + N_f(s)} = \frac{6(5+s)}{(s+3)+6(5+s)} = \frac{6(s+5)}{(7s+33)} = \left(\frac{6}{7}\right)\frac{s+5}{s+33/7}$$

so that the closed -loop pole is at $s = -33/7 = -4.7143$ and the closed-loop zero is at $s = -5$ (the same as the open loop zero defined by the P-D controller).

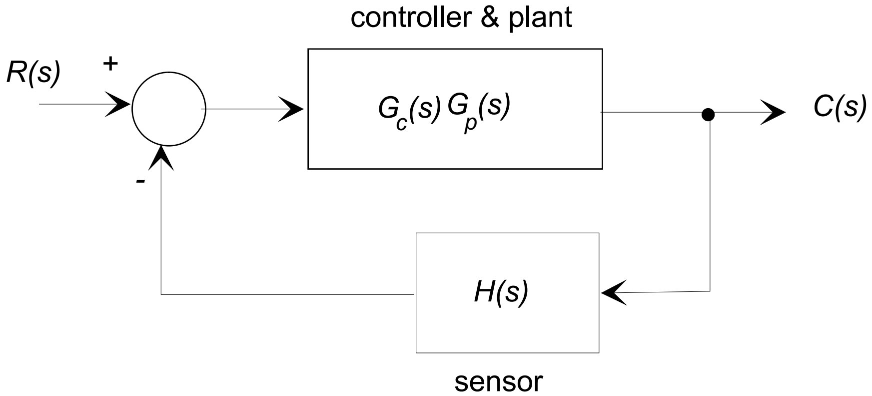
Aside: The system can be analyzed using the following MATLAB commands:

```
forward_system = zpk(-5, -3, 6)
closed_loop =feedback(forward_system,1)
pole(closed_loop)   %Find the closed-loop system poles
zero(closed_loop)   %Find the closed-loop system system zeros
pzmap(closed_loop)  %Make a pole-zero plot
step(closed_loop)   %Plot the closed-loop step response.
```



The step response is shown below – note the initial transient caused by the direct feed-through.

Now consider a closed-loop system with sensor dynamics $H(s)$



The closed-loop transfer function is

$$C_{cl}(s) = \frac{G_f(s)}{1 + G_f(s)H(s))} = \frac{N_f(s)D_H(s)}{D_f(s)D_H(s) + N_f(s)N_H(s)}$$

where $N_H(s)$ and $D_H(s)$ are the numerator and denominator polynomials of the sensor transfer function $H(s)$. In this case:

- The closed-loop poles are the roots of the characteristic equation $D_f(s)D_H(s) + N_f(s)N_H(s) = 0$.
- The closed-loop zeros are the *zeros* of the forward transfer function, and the *poles* of the sensor transfer function.

■ Example 2

Repeat the previous example with a sensor that has a transfer function $H(s) = 10/(s + 10)$. The forward transfer function is

$$G_f(s) = G_c(s)G_p = \frac{6(5 + s)}{s + 3}$$

and

$$H(s) = \frac{10}{s + 10}$$

The closed-loop transfer function is:

$$G_{cl}(s) = \frac{N_f(s)D_H(s)}{D_f(s)D_H(s) + N_f(s)N_H(s)} = \frac{6(5 + s)(s + 10)}{(s + 10)(s + 3) + 60(5 + s)} = \frac{6(s^2 + 15s + 50)}{(s^2 + 73s + 330)}$$

so that the closed-loop poles are at $s = -4.84, -68.16$ and the closed-loop zeros are at $s = -5, -10$ (the same as the open loop zero defined by the P-D controller, and the pole associated with the sensor).

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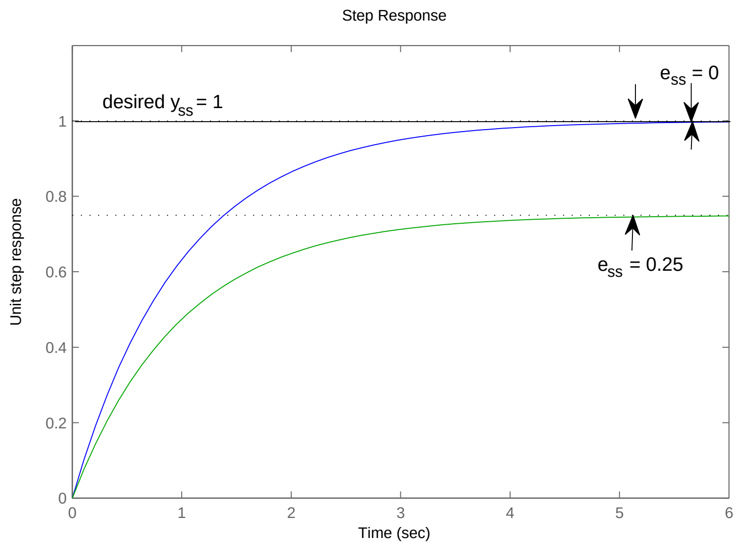
2 Steady-State Errors

In the lab we have considered steady-state errors for both velocity and position control of the rotary inertia, and we have noted:

- There was a finite s.s. error with a constant input under *velocity* control.
- That the s.s. error was eliminated when we used PI (proportional + integral) control.
- There was no s.s. error with a constant input for *position* control.

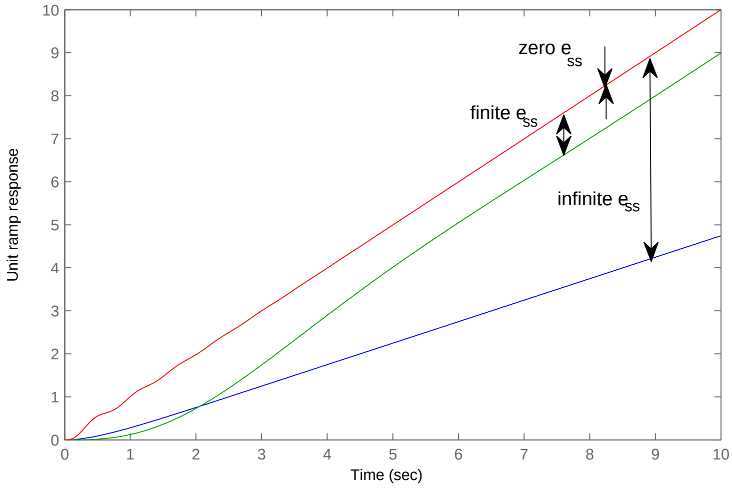
We will look at the steady state error for two basic inputs

1. The step input. The step response measures the ability of a feedback control system to regulate the output to a *constant input*.

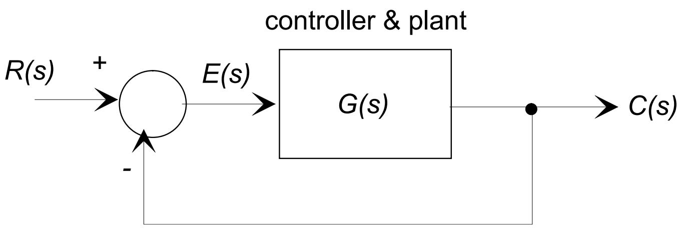
   The above figure shows the response to a unit step. One response exhibits $e_{ss} = 0$, while the other shows a constant steady-state error.

2. The ramp input. The steady-state ramp response error is a measure of a feedback control system's ability to follow a simple time-varying *trajectory*.



The above figure shows three responses to a unit ramp input $r(t) = t$. In one case there is no steady-state error - as t becomes large, the response follows the input exactly. In the second case there is a finite steady state error, the response has unit slope but exhibits a constant offset from the input. The third case shows a response in which the error is growing without bound, and the steady-state error is infinite.

We now look at the whole question of steady-state errors under closed-loop control, and methods to eliminate them. Consider the unity feedback system:



The error signal $E(s)$ is defined to be $E(s) = R(s) - C(s)$, and the transfer function relating the error to the input command is found from:

$$\begin{aligned} C(s) &= G)s)E(s) \\ E(s) &= R(s) - C(s) \end{aligned}$$

giving

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{D(s)}{D(s) + N(s)}$$

where $N(s)$ and $D(s)$ are the numerator and denominator polynomials of $G(s)$ respectively.

We recall the final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

so that

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0}\left(sR(s)\frac{D(s)}{D(s)+N(s)}\right)$$

Now consider the two cases:

Step input: In this case, when $r(t) = u_s(t)$, $R(s) = 1/s$ so that

$$e_{ss} = \lim_{s\to 0}\left(s\frac{1}{s}\frac{D(s)}{D(s)+N(s)}\right) = \lim_{s\to 0}\left(\frac{D(s)}{D(s)+N(s)}\right)$$

The condition to ensure that $e_{ss} = 0$ therefore must be that

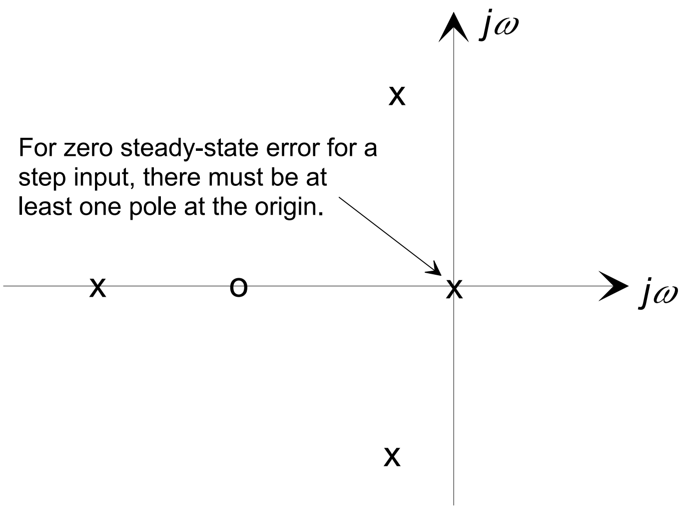
$$\lim_{s\to 0} D(s) = 0$$

If

$$G(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

or $D(s) = \prod_{i=1}^{n}(s-p_i)$, to ensure that $\lim_{s\to 0} D(s) = 0$ we require that at least one of the $p_i = 0$ (one or more poles of the system be at the origin). This is equivalent to saying that the forward transfer function must be of the form

$$G(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{s^k\prod_{i=k+1}^{n}(s-p_i)} \qquad k \ge 1$$



Ramp input: In this case, when the input is a unit ramp $r(t) = t$, $R(s) = 1/s^2$

$$e_{ss} = \lim_{s\to 0}\left(s\frac{1}{s^2}\frac{D(s)}{D(s)+N(s)}\right) = \lim_{s\to 0}\left(\frac{1}{s}\frac{D(s)}{D(s)+N(s)}\right)$$

The condition to ensure that $e_{ss} = 0$ therefore must be that

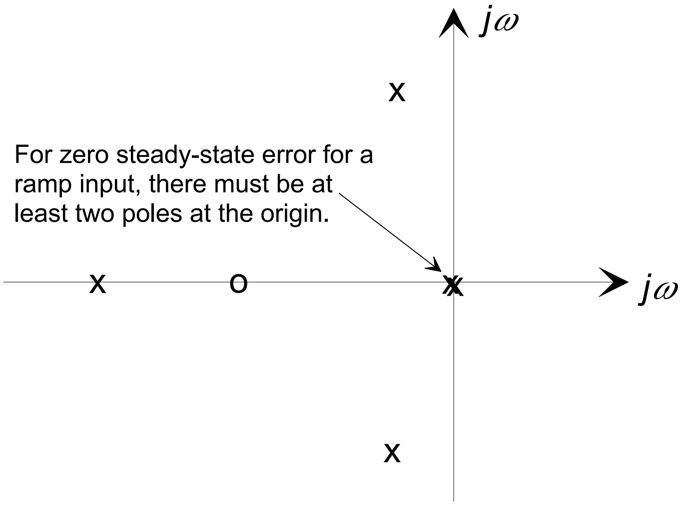
$$\lim_{s\to 0}\frac{D(s)}{s} = 0$$

If as before

$$G(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)}$$

or $D(s) = \prod_{i=1}^{n}(s-p_i)$, to ensure that $\lim_{s\to 0} D(s)/s = 0$ we require that at least two of the $p_i = 0$ (one or more poles of the system be at the origin). This is equivalent to saying that the forward transfer function must be of the form

$$G(s) = K\frac{\prod_{i=1}^{m}(s-z_i)}{s^k\prod_{i=k+1}^{n}(s-p_i)} \qquad k \geq 2.$$



The above argument can be extended as follows:

For zero steady-state error to a waveform with a Laplace transform $1/s^k$, the forward transfer function must have at least k poles at the origin.

2.1 System Type

Poles at the origin $s = 0$ are known as *free integrators*. The *System Type* is defined as the number of free integrators in the system.

Type 0: - $$G(s) = K\frac{(s-z_1)\dots(s-z_m)}{(s-p_1)\dots(s-p_n)}$$

Type 1: - $$G(s) = K\frac{(s-z_1)\dots(s-z_m)}{s(s-p_2)\dots(s-p_n)}$$

Type 2: - $$G(s) = K\frac{(s-z_1)\dots(s-z_m)}{s^2(s-p_1)\dots(s-p_n)}$$

and we can say

- For a system under proportional control, to ensure $e_{ss} = 0$ for a step input, the system must be at least Type 1.

- For a system under proportional control, to ensure $e_{ss} = 0$ for a ramp input, the system must be at least Type 2.

and we can make the following table showing the steady-state error conditions:

	Type 0	Type 1	Type 2	Type 3
step	e_{ss} = constant	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = 0$
ramp	$e_{ss} = \infty$	e_{ss} = constant	$e_{ss} = 0$	$e_{ss} = 0$
parabola	$e_{ss} = \infty$	$e_{ss} = \infty$	e_{ss} = constant	$e_{ss} = 0$

■ Example 3

In the lab you (should have) observed that with proportional control (1) that the velocity control gave a finite steady state error for a constant input, whereas (2) the position control had zero steady-state error.

For velocity control:

$$G(s) = \frac{\Omega(s)}{\Omega_d} = \frac{K_p}{Js + B}$$

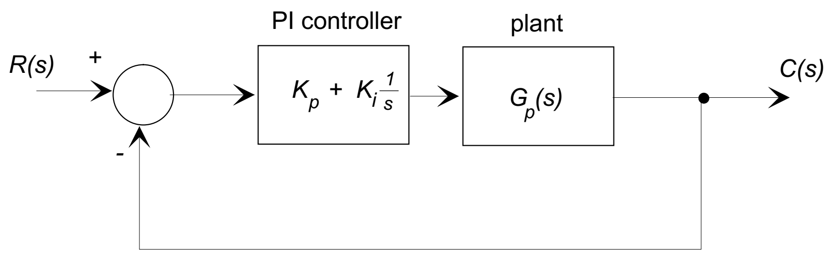
which is a Type 0 system, which will have a finite steady-state error. For position control:

$$G(s) = \frac{\theta(s)}{\theta_d(s)} = \frac{K_p}{s(Js + B)}$$

which is a Type 1 system, and from the above argument will have a zero steady-state error.

■ Example 4

Show why PI control reduces the steady-state error to zero for a step input with a Type 0 system.



For a PI controller, the transfer function is

$$G_c(s) = K_p + K_i \frac{1}{s} = \frac{K_p s + K_i}{s}$$

so that the controller introduces (1) a pole at the origin, and(2) a zero at $s = -K_i/K_p$ so that the forward transfer function is

$$G_f(s) = G_c(s)G_p(s) = K_p \frac{(s + K_i/K_p)}{s} G_p(s)$$

which is Type 1, and will have zero steady-state error for a constant input.

2.2 Static Error Constants

Recall that the transfer function relating the error to the input is

$$E(s)\frac{1}{1 + G(s)}.$$

For a step input

$$e_{ss} = \lim_{s \to 0} s \left(\frac{1}{s}\right) \frac{1}{1 + G(s)} = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

If we define a static position constant K_p (not to be confused with a controller gain) as

$$\boxed{K_p = \lim_{s \to 0} G(s) \quad \text{then} \quad e_{ss} = \frac{1}{1 + K_p}}$$

Similarly, for a ramp input (constant velocity)

$$e_{ss} = \lim_{s \to 0} s \left(\frac{1}{s^2}\right) \frac{1}{1 + G(s)} = \frac{1}{\lim_{s \to 0} sG(s)}$$

and, if we define a static velocity constant K_v as

$$\boxed{K_v = \lim_{s \to 0} sG(s) \quad \text{then} \quad e_{ss} = \frac{1}{K_v}.}$$

We can also define an acceleration constant K_a for parabolic inputs, since

$$e_{ss} = \lim_{s \to 0} s \left(\frac{1}{s^3}\right) \frac{1}{1 + G(s)} = \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

so that if

$$\boxed{K_a = \lim_{s \to 0} s^2 G(s) \quad \text{then} \quad e_{ss} = \frac{1}{K_a}.}$$

Input	Error	Type 0	Type 1	Type 2
step	$1/(1 + K_p)$	$e_{ss} = 1/(1 + K_p)$	$K_p = \infty$, $e_{ss} = 0$	$K_p = \infty$, $e_{ss} = 0$
ramp	$1/K_v$	$K_v = 0$, $e_{ss} = \infty$	$e_{ss} = 1/K_v$	$K_v = \infty$, $e_{ss} = 0$
parabola	$1/K_a$	$K_v = 0$, $e_{ss} = \infty$	$K_v = 0$, $e_{ss} = \infty$	$1/K_a$