2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Thomas Peacock
3/7/2007

Lecture 9

2D Motion of Rigid Bodies: Kinetics, Poolball Example

Kinetics of Rigid Bodies

Angular Momentum Principle for a Rigid Body

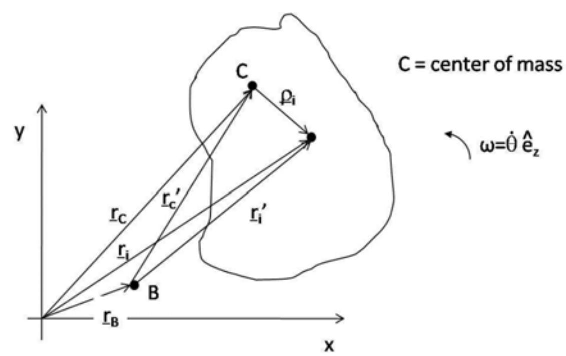

Figure 1: Rigid Body rotating with angular velocity ω. Figure by MIT OCW.

$$\underline{H}_B = \sum_i r_i' \times m_i(\underline{v}_c + \underline{\omega} \times \underline{\rho}_i)$$

After some steps (see Lecture 8):

$$\underline{H}_B = \underline{r}_c' \times \underline{P} + \sum_i m_i \underline{\rho}_i \times \underline{\omega} \times \underline{\rho}_i$$

We now use:

$$\underline{a} \times \underline{b} \times \underline{c} = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$$

$$\begin{aligned}\underline{\rho}_i \times \underline{\omega} \times \underline{\rho}_i &= \rho_i^2 \underline{\omega} - (\underline{\omega} \cdot \underline{\rho}_i)\underline{\rho}_i \\ &= \rho_i^2 \underline{\omega}\end{aligned}$$

For 2-D motion, $\underline{\omega} \cdot \underline{\rho}_i = 0$ because the vectors are $\perp$. For 3-D, this term does not have to be 0.

$$\begin{aligned}\underline{H}_B &= \underline{r}_c^{'} \times \underline{P} + \sum_i m_i \underline{\rho}_i^2 \underline{\omega} \\ &= \underline{r}_c^{'} \times \underline{P} + I_c \underline{\omega}\end{aligned}$$

I_c: Moment of Inertia. $I_c = \sum_i m_i \rho_i^2$ (Intrinsic Property of Rigid Body)

Example:

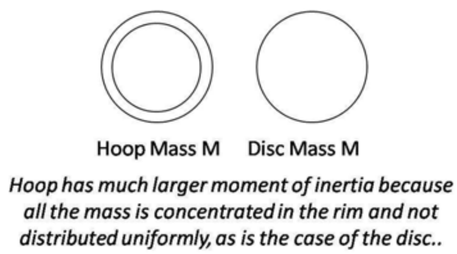

Figure 2: Hoop and Disc, both with mass M. Figure by MIT OCW.

$$\boxed{\underline{H}_B = \underline{r}_c^{'} \times \underline{P} + I_c \underline{\omega}}$$

If one takes angular momentum about the center of mass:

$$\boxed{\underline{H}_c = I_c \underline{\omega}}$$

(Angular Momentum about B) = (Angular Momentum about C) + (Moment of Linear Momentum about B)

Therefore:

$$\boxed{\underline{H}_B = \underline{H}_c + \underline{r}_c^{'} \times \underline{P}}$$

Special Case of Fixed Axis of Rotation about B

i.e. $\underline{v}_c = \underline{v}_B + \underline{\omega} \times \underline{r}_c^{'}$

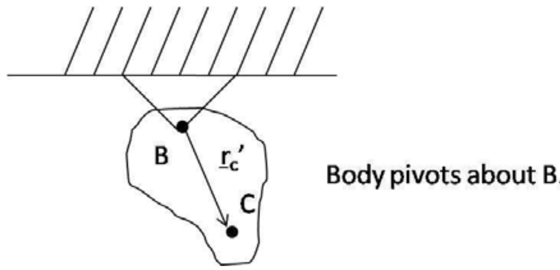

Figure 3: Rigid body which pivots about B. Figure by MIT OCW.

$$\begin{aligned}\underline{H}_B &= \underline{H}_C + r_c^{'} \times m(\underline{\omega} \times \underline{r}_c^{'}) \\ &= \underline{H}_C + m r_C^{'2} \underline{\omega} \\ &= (I_C + m r_C^{'2})\underline{\omega} = I_B \underline{\omega}\end{aligned}$$

$$\boxed{I_B = I_C + m r_C^{'2} \qquad \text{Parallel Axis Theorem}}$$

$$\boxed{\text{Only do this if the } \underline{v}_B = 0 \text{ and } \underline{v}_C = (\underline{\omega} \times r_C^{'})}$$

Finally:

$$\tau_B^{ext} = \frac{d}{dt} H_B + \underline{v}_B \times \underline{P} \tag{1}$$

$$\underline{H}_B = \underline{H}_C + \underline{r}_C^{'} \times \underline{P} \tag{2}$$

$$\underline{H}_C = I_C \underline{\omega} \tag{3}$$

$$I_C = \sum m_i \rho_i^2 \tag{4}$$

Equations (1) to (4) are always true.

Special Cases

1. $B = C \Rightarrow \underline{r}_C^{'} = 0; \underline{v}_B \parallel \underline{P}$

Start by thinking about motion around center of mass.

$$\tau_B^{ext} = \frac{d}{dt} \underline{H}_C \text{ and } \underline{H}_C = I_C \underline{\omega}$$

2. B is a stationary point and fixed in the body.

$$\tau_B^{ext} = \frac{d}{dt}\underline{H}_B \text{ and } \underline{H}_B = I_B\underline{\omega} \text{ where } I_B = I_C + mr_C'^2$$

What do we need to do still?
 Calculating moments of inertia ⇒ Recitation 5
 Work-Energy Principle

Cue hitting a pool ball

A pool ball of radius R and mass M is at rest on a horizontal table. It is set in motion by a sharp horizontal impulse $\underline{J}$ provided by the cue. Determine the height above the ball's center that the cue should strike so that the subsequent motion is rolling without slipping.

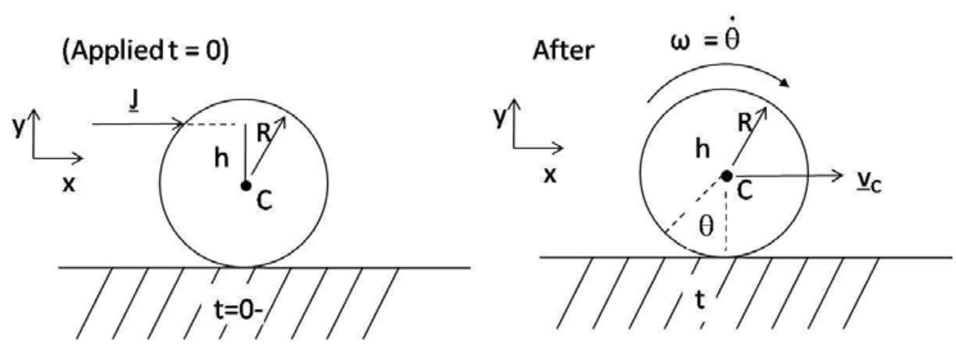

Figure 4: Cue ball diagram. Diagram shows cue ball when force if first applied and after being hit. Figure by MIT OCW.

Hit below h: Backspin
Hit above h: Top spin, carry on shot

Kinematics: Geometry with no forces

Horizontal Table: y_C =constant = R
Rolling without slipping: $\underline{v}_C = \underline{\omega}R$ (or $x_c = R\theta$)
1 Degree of Freedom. (Use x_C or θ).

Kinetics: Free Body Diagrams

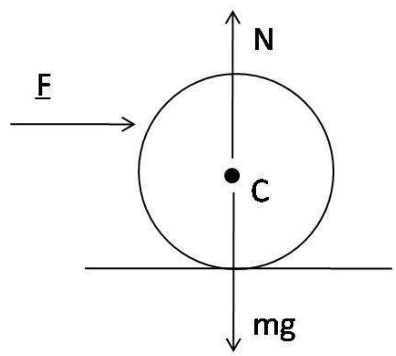

Figure 5: Free Body Diagram of Cue Ball. Figure by MIT OCW.

Impulse force that provides impulse $\underline{J}$

$$\underline{J} = \int_{0^-}^{0^+} \underline{F} dt = \int_{0^-}^{0^+} \underline{J}\delta(t) dt \text{ i.e. } \underline{F} = \underline{J}\delta(t)$$

(i) Linear Momentum Principle

$$\underline{F}^{ext} = \frac{d}{dt}\underline{P}$$

y-direction: C always at same height. $N = mg$ so no vertical motion of C.
x-direction: $F = J\delta(t) = \frac{d}{dt} M v_C$.
Integrate both sides

$$\int_{0^-}^{0^+} F dt \int_{0^-}^{0^+} J\delta(t) dt = \int_{0^-}^{0^+} \frac{d}{dt} M v_c dt$$
$$J = M v_c(0^+) - M v_c(0^-)$$

J: Momentum Imparted

$$\boxed{J = M v_C(0^+)} \quad (5)$$

Angular Momentum Principle About C

Taking momentum about C simplifies equations

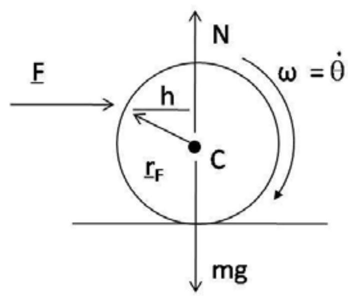

Figure 6: Angular Momentum Principle about C applied to Cue Ball. Figure by MIT OCW.

$$\underline{\tau}_C^{ext} = \frac{d}{dt}\underline{H}_c \text{ and } \underline{H}_C = I_C\underline{\omega}$$

$$\underline{r}_F \times \underline{F} = \frac{d}{dt}I_C\underline{\omega}$$

$$-Fh\hat{e}_z = -I_C\frac{d\underline{\omega}}{dt}\hat{e}_z$$

$$\int_{0^-}^{0^+} \underline{F}h dt = \int_{0^-}^{0^+} I_C\frac{d\omega}{dt}dt$$

$$\int_{0^-}^{0^+} J\delta(t)dt = I_C\omega(0^+) - I_C\omega(0^-)$$

$I_C\omega(0^-) = 0$ because $\omega(0^-) = 0$.

$$\boxed{Jh = I_C\omega(0^+)}$$

Impulsive torque about center of mass = Change of angular momentum caused by the torque

Satisfying Constraints

If there is no slip, one needs $\omega(0^+)R = v_C(0^+)$

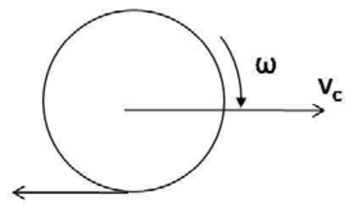

Figure 7: Diagram of Cue Ball moving. This diagram demonstrates how to satisfy geometric constraints of movement. Figure by MIT OCW.

$$J = \frac{I_C}{h}\frac{v_C(0^+)}{R} \tag{6}$$

Can eliminate J from Equation 5 and Equation 6.

$$Mv_C(0^+) = \frac{I_C}{h}\frac{v_C(0^+)}{R} \Rightarrow \boxed{h = \frac{I_C}{mR}}$$

For a sphere:

$$I_C = \frac{2}{5}mR^2 \Rightarrow \boxed{h = \frac{2}{5}R}$$

h: Independent of mass of sphere. Independent of force applied.