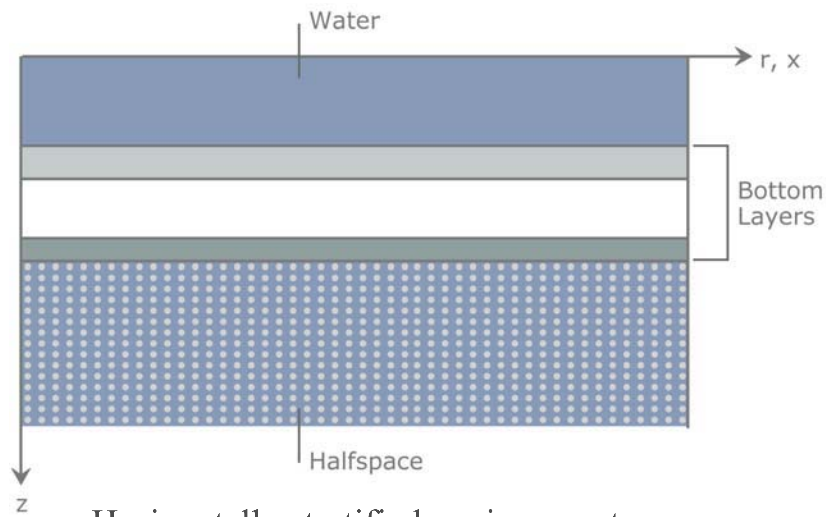
Ocean Acoustic Theory

- Acoustic Wave Equation
- Integral Transforms
- Helmholtz Equation
- Source in Unbounded and Bounded Media
- Propagation in Layered Media
  - Reflection and Transmission
- The Ideal Waveguide
  - Image Method
  - Wavenumber Integral
  - Normal Modes
- Pekeris Waveguide



Horizontally stratified environment

Layered Media and Waveguides

Integral Transform Solution

Helmholtz Equation - Layer n

$$\left[\nabla^2 + k_n^2(z)\right] \psi(\mathbf{r}) = f(\mathbf{r}) \, ,$$

Interface Boundary Conditions

$$B\left[\psi(\mathbf{r})\right]|_{z=z_n} = 0 \, , \quad n = 1 \cdots N \, ,$$

Axisymmetric Propagation Problems: Hankel Transform Solution

$$
\begin{aligned}
f(r,z) &= \int_0^\infty f(k_r,z)\, J_0(k_r r)\, k_r\, dk_r \,, \\
f(k_r,z) &= \int_0^\infty f(r,z)\, J_0(k_r r)\, r\, dr \,,
\end{aligned}
$$

Depth-Separated Wave Equation

$$
\left[\frac{d^2}{dz^2} + (k^2 - k_r^2)\right] \psi(k_r,z) = S_\omega \, \frac{\delta(z-z_s)}{2\pi} \,.
$$

Superposition Pr

Depth-Separated Green's Function

$$
\psi(k_r,z) = -S_\omega \, G_\omega(k_r,z,z_s) = -S_\omega \left[g_\omega(k_r,z,z_s) + H_\omega(k_r,z) \right]
$$

Source contribut

$$
\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] g_\omega(k_x,z,z_s) \;=\; -\frac{\delta(z-z_s)}{2\pi}
$$

Homogeneous S

$$
\left[\frac{d^2}{dz^2} + (k^2 - k_x^2)\right] H_\omega(k_x,z) \;=\; 0
$$

Interface Boundary Conditions

$$
B\left[\psi(k_r,z_n)\right] = 0 \,.
$$

Source field

$$
\begin{aligned}
g_\omega(k_r, z, z_s) &= A(k_r) \begin{cases} e^{ik_z(z-z_s)}\,, & z \geq z_s \\ e^{-ik_z(z-z_s)}\,, & z \leq z_s \end{cases} \\
&= A(k_r)\, e^{ik_z|z-z_s|}\,.
\end{aligned}
$$

Integration of depth-separated wave equation over $[z_s - \epsilon, z_s + \epsilon]$:

$$
\left[\frac{dg_\omega(k_r, z)}{dz}\right]_{z_s-\epsilon}^{z_s+\epsilon} + O(\epsilon) = -\frac{1}{2\pi}\,.
$$

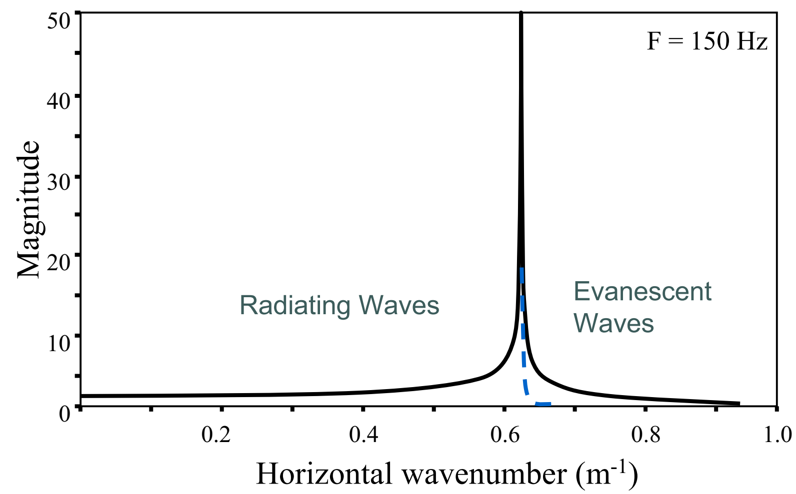
$$
\begin{aligned}
&\Rightarrow \\
A(k_r) &= -\frac{1}{4\pi i k_z} \\
&\Rightarrow \\
g_\omega(k_r, z, z_s) &= -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z}\,.
\end{aligned}
$$

Inverse Hankel Transform - Sommerfeld–Weyl Integral

$$
g_\omega(r, z, z_s) = \frac{i}{4\pi} \int_0^\infty \frac{e^{ik_z|z-z_s|}}{k_z} J_0(k_r r)\, k_r\, dk_r\,,
$$

Grazing Angle Representation

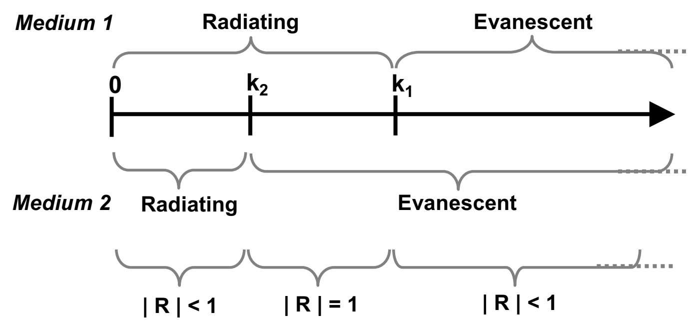
$$
\begin{aligned}
k_x &= k \cos\theta\,, \\
k_z &= k \sin\theta\,, \\
\frac{dk_x}{d\theta} &= -k_z\,.
\end{aligned}
$$



Magnitude of the depth-dependent Green's function for point source in an infinite medium. Solid curve: $z - z_s = \lambda/10$; dashed curve: $z - z_s = 2\,\lambda$.

$$
\begin{aligned}
&\Rightarrow \\
g_\omega(\mathbf{r}, \mathbf{r}') &\simeq \frac{i}{4\pi} \int_{-k}^{k} \frac{e^{ik_z|z-z_s|}}{k_z} e^{ik_x x}\, dk_x \\
&= \frac{i}{4\pi} \int_0^\pi e^{ik\,|z-z_s|\sin\theta + ikx\cos\theta}\, d\theta\,.
\end{aligned}
$$

Example: Hard Bottom – $c_2 > c_1$



Spectral domains for a hard bottom, $k_2 < k_1$.

1. $k_r < k_2$: Waves are *propagating* vertically in both media and energy will be transmitted into the bottom: $|R| < 1$.
2. $k_2 < k_r < k_1$: Waves are *propagating* in the upper halfspace (water) but are *evanescent* in the lower halfspace (bottom): $|R| = 1$.
3. $k_1 < k_r$: Waves are *evanescent* in depth in both media: $|R| < 1$.

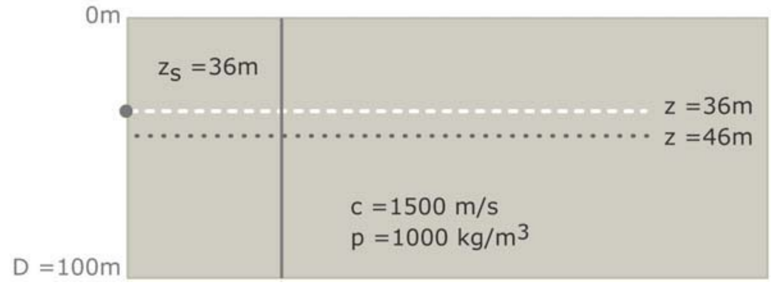
Magnitude and Phase

$$R(\theta) = |R(\theta)|\, e^{-i\phi(\theta)} ,$$

Critical Angle

$$\theta_c = \arccos\left(k_2/k_1\right)$$

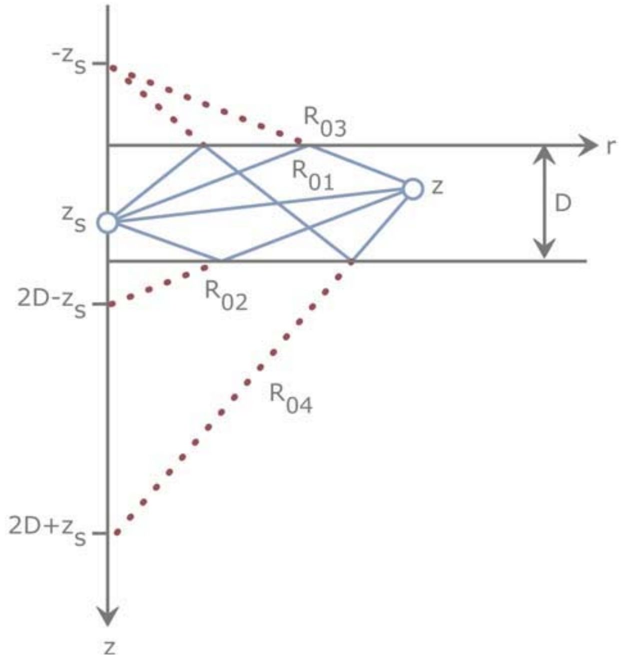
[See Fig 2.10 in Jensen, Kuperman, Porter and Schmidt. *Computational Ocean Acoustics.* New York: Springer-Verlag, 2000.]



Idealized ocean waveguide model with pressure-release surface and bottom

Ideal Fluid Waveguide

Image Method



Ray Expansion

$$\psi(r,z) = -\frac{S_\omega}{4\pi}\sum_{m=0}^{\infty}\left[\frac{e^{ikR_{m1}}}{R_{m1}} - \frac{e^{ikR_{m2}}}{R_{m2}} - \frac{e^{ikR_{m3}}}{R_{m3}} + \frac{e^{ikR_{m4}}}{R_{m4}}\right],$$

Ideal Fluid Waveguide

Integral Transform Solution

Wavenumber Integral Representation

$$\psi(r,z) = \int_0^\infty \psi(k_r,z)\, J_0(k_r r)\, k_r\, dk_r \ ,$$

Superposition Principle

$$\psi(k_r,z) = -S_\omega\ [g_\omega(k_r,z,z_s) + H_\omega(k_r,z)]\ .$$

$$g_\omega(k_r,z,z_s) = -\frac{e^{ik_z|z-z_s|}}{4\pi i k_z}\ ,$$

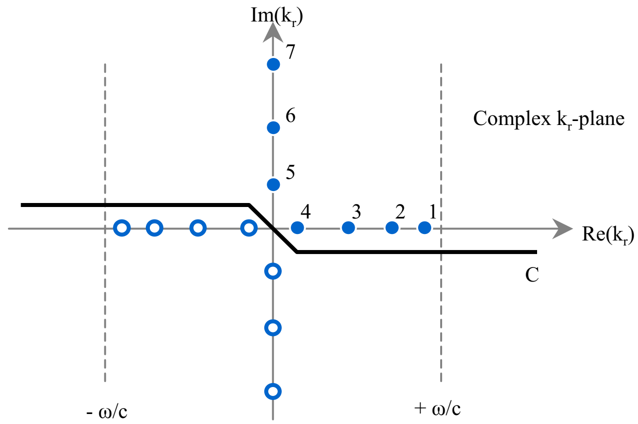
$$H_\omega(k_r,z) = A^+(k_r)\, e^{ik_z z} + A^-(k_r)\, e^{-ik_z z}\ .$$

Pressure Release Boundary Conditions at $z = 0, D$

$$\begin{aligned} A^+(k_r) + A^-(k_r) &= \frac{e^{ik_z z_s}}{4\pi i k_z} \\ A^+(k_r)\, e^{ik_z D} + A^-(k_r)\, e^{-ik_z D} &= \frac{e^{ik_z(D-z_s)}}{4\pi i k_z}\ . \end{aligned}$$

Wavenumber Kernel of Total Field

$$\psi(k_r,z) = -\frac{S_\omega}{4\pi} \begin{cases} \frac{\sin k_z z\ \sin k_z(D-z_s)}{k_z\ \sin k_z D}, & z < z_s \\ \frac{\sin k_z z_s\ \sin k_z(D-z)}{k_z\ \sin k_z D}, & z > z_s\ . \end{cases}$$



Singularities of the depth-dependent Green's function for an ideal waveguide.

Ideal Fluid Waveguide

Poles of Wavenumber Kernel

$$k_z D = m\pi\,, \qquad m = 1, 2\ldots\,,$$

or, in terms of the horizontal wavenumber $k_r = \sqrt{k^2 - k_z^2}\,$,

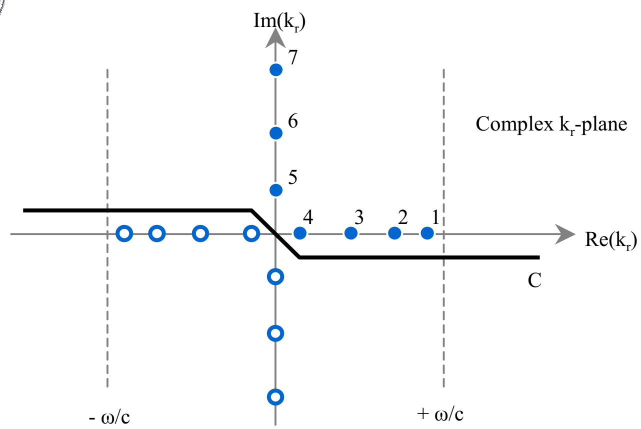
$$k_r = \sqrt{k^2 - \left(\frac{m\pi}{D}\right)^2}\,, \qquad m = 1, 2\ldots$$

Field Integral Representation

$$\psi(r, z) = \frac{1}{2}\int_{-\infty}^{\infty} \psi(k_r, z)\, H_0^{(1)}(k_r r)\, k_r\, dk_r\ .$$

Field Evaluation Techniques

1. *Method of Stationary Phase:* Ray Methods
2. *Numerical Integration:* Wavenumber Integration approach.
3. *Contour integration by residuals:* Normal Modes.



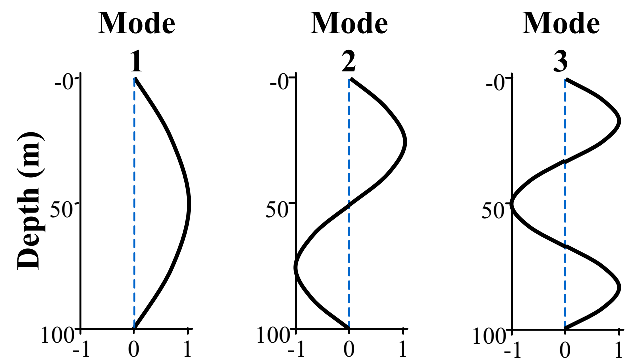
Singularities of the depth-dependent Green's function for an ideal waveguide.

Ideal Fluid Waveguide

Normal Modes

$$\psi(r,z) = -\frac{iS_\omega}{2D} \sum_{m=1}^{\infty} \sin(k_{zm}z)\,\sin(k_{zm}z_s)\; H_0^{(1)}(k_{rm}r)\;,$$

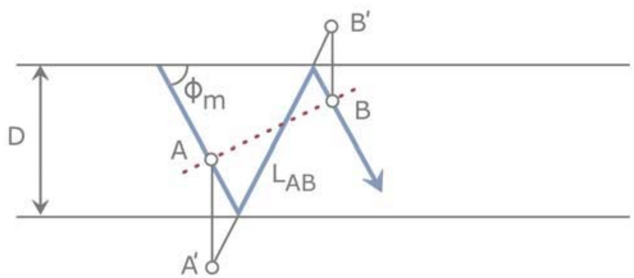
$$\begin{array}{llll} \textit{Propagating modes:} & k_{rm} & \text{real} & m < \frac{kD}{\pi}\;, \\ \textit{Evanescent modes:} & k_{rm} & \text{imaginary} & m > \frac{kD}{\pi}\;. \end{array}$$



Depth dependence of the first 3 normal modes in ideal waveguide at 20Hz.

Modes as Superposition of Plane Waves

$$\sin(k_{zm}z) = \frac{e^{ik_{zm}z} - e^{-ik_{zm}z}}{2i}\;.$$



$$\theta_m = \arctan\left(k_{zm}/k_{rm}\right)$$

$$L_{AB} = \frac{2D}{\sin\theta_m} - \frac{2D}{\tan\theta_m}\cos\theta_m = 2D\sin\theta_m = \frac{2\pi m}{k} = m\lambda\;,$$

Modal Dispersion

$$
\begin{aligned}
k_r &= \sqrt{(\omega/c)^2 - \left(\frac{m\pi}{D}\right)^2} \\
&\Leftrightarrow \\
\omega &= c\sqrt{k_{rm}^2 + \left(\frac{m\pi}{D}\right)^2} .
\end{aligned}
$$

Modal Cut-off Frequencies

$$f_{0m} = \frac{\omega_{0m}}{2\pi} = \frac{mc}{2D} ,$$

Modal Phase Velocity

$$v_m = \frac{\omega}{k_{rm}} .$$

[see Jensen, Fig 2.21]

Modal Group Velocity

$$\psi(t) = \int_{\omega-\epsilon}^{\omega+\epsilon} \psi(\omega)\, e^{-i[\omega t - k_{rm}(\omega)\, r]}\, d\omega .$$

$$
\begin{aligned}
d\omega\, dt - dk_{rm}(\omega)\, dr &= 0 \\
&\Leftrightarrow \\
u_m = \frac{dr}{dt} &= \frac{d\omega}{dk_{rm}}
\end{aligned}
$$

[see Jensen, Fig 2.22]

Ideal Fluid Waveguide

The Waveguide Field

$$
\begin{aligned}
|\psi(r,z)| &\simeq r^{-1/2}\left|A_m\, e^{ik_{rm}r} + A_n\, e^{ik_{rn}r}\right| \\
&= r^{-1/2}\sqrt{A_m^2 + A_n^2 + 2A_mA_n\cos\left[r\left(k_{rm}-k_{rn}\right)\right]}\,.
\end{aligned}
$$

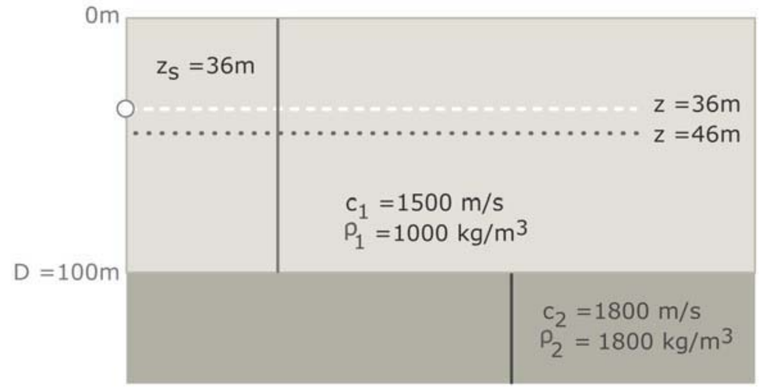
Amplitude Oscillation Period - Modal Interference Length

[see Jensen, Figs 2.23 & 2.24]

$$
L = \frac{2\pi}{k_{rm}-k_{rn}}\,,
$$

Propagating Modes at 20 Hz

$$
M < \frac{kD}{\pi} = \frac{2fD}{c} = 2.6667\,.
$$



Pekeris waveguide with pressure-release surface and penetrable fluid bottom

The Pekeris Waveguide

Field in Water

$$\psi_1(k_r, z) = S_\omega \frac{e^{ik_{z,1}|z-z_s|}}{4\pi i k_{z,1}} + A_1^+(k_r)\, e^{ik_{z,1}z} + A_1^-(k_r)\, e^{-ik_{z,1}z} \,,$$

Field in Bottom

$$\psi_2(k_r, z) = A_2^+(k_r)\, e^{ik_{z,2}(z-D)} \,,$$

Vertical Wavenumber

$$k_{z,2} = \begin{cases} \sqrt{k_2^2 - k_r^2}\,, & |k_r| < k_2 \\ i\sqrt{k_r^2 - k_2^2}\,, & |k_r| > k_2 \,, \end{cases}$$

Interface Conditions

Surface Pressure Release

$$A_1^+(k_r) + A_1^-(k_r) = S_\omega \, \frac{ie^{ik_{z,1} z_s}}{4\pi k_{z,1}} \, .$$

Seabed Displacement Continuity

$$k_{z,1} \, e^{ik_{z,1}D} \, A_1^+(k_r) - k_{z,1} \, e^{-ik_{z,1}D} \, A_1^-(k_r) - k_{z,2} \, A_2^+(k_r) = k_{z,1} \, S_\omega \, \frac{ie^{ik_{z,1}(D-z_s)}}{4\pi k_{z,1}},$$

Seabed Pressure Continuity

$$\rho_1 \, e^{ik_{z,1}D} \, A_1^+(k_r) + \rho_1 \, e^{-ik_{z,1}D} \, A_1^-(k_r) - \rho_2 \, A_2^+(k_r) = \rho_1 S_\omega \, \frac{ie^{ik_{z,1}(D-z_s)}}{4\pi k_{z,1}} \, .$$

Global Matrix Equations

$$\begin{bmatrix} 1 & 1 & 0 \\ k_{z,1} \, e^{ik_{z,1}D} & -k_{z,1} \, e^{-ik_{z,1}D} & -k_{z,2} \\ \rho_1 \, e^{ik_{z,1}D} & \rho_1 \, e^{-ik_{z,1}D} & -\rho_2 \end{bmatrix} \begin{Bmatrix} A_1^+ \\ A_1^- \\ A_2^+ \end{Bmatrix} = \frac{iS_\omega}{4\pi k_{z,1}} \begin{Bmatrix} e^{ik_{z,1}z_s} \\ k_{z,1} \, e^{ik_{z,1}(D-z_s)} \\ \rho_1 \, e^{ik_{z,1}(D-z_s)} \end{Bmatrix} .$$

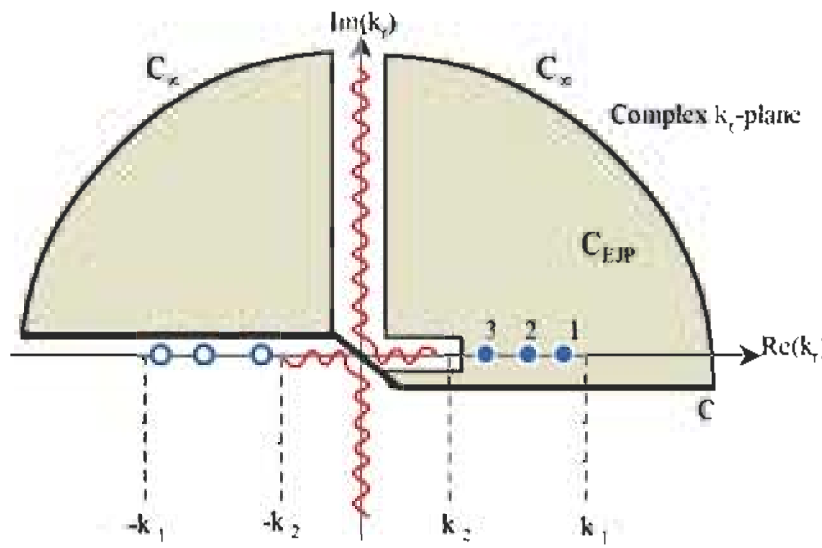
Normal Modes

Normal Modes

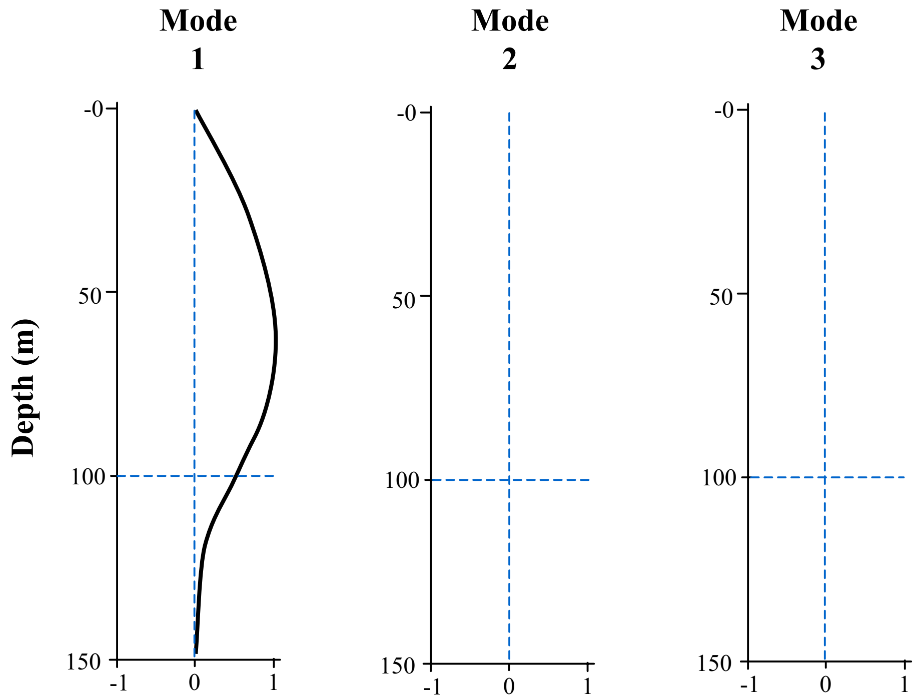
$$\det(k_r) = -2i\left[\rho_1 k_{z,2}\sin(k_{z,1}D) + i\rho_2 k_{z,1}\cos(k_{z,1}D)\right]$$

$$\tan(k_{z,1}D) = -\frac{i\rho_2 k_{z,1}}{\rho_1 k_{z,2}}\,.$$

Propagating modes: $|k_2| < |k_r| < |k_1|$.



Complex wavenumber plane with EJP branch cut, poles, and integration contour.



Depth dependence of acoustic pressure for the 3 normal modes in the Pekeris waveguide at 35Hz.

Modal Expansion

$$\psi(r,z) \simeq -\frac{iS_\omega}{2D}\sum_{m=1}^{M} a_m(k_{rm})\sin(k_{zm}z)\sin(k_{zm}z_s)\,H_0^{(1)}(k_{rm}r)\ ,$$

Modal Dispersion

Modal Expansion

$$\psi(r,z) \simeq -\frac{iS_\omega}{2D} \sum_{m=1}^{M} a_m(k_{rm}) \sin(k_{zm} z) \sin(k_{zm} z_s)\, H_0^{(1)}(k_{rm} r)\,,$$

Modal Dispersion

Modal Cut-off Frequencies

$$\begin{aligned}
k_{z,2} &= 0 \\
&\Leftrightarrow \\
k_{zm} D &= \omega_{0m} D \sqrt{c_1^{-2} - c_2^{-2}} \\
&= \frac{\pi}{2} + (m-1)\,\pi, \qquad m = 1, 2, \cdots \\
&\Rightarrow \\
f_{0m} &= \frac{\omega_{0m}}{2\pi} \\
&= \frac{(m-0.5)\, c_1 c_2}{2D\sqrt{c_2^2 - c_1^2}}\,. \qquad m = 1, 2, \cdots
\end{aligned}$$

[see Jensen, Fig 2.28]

High-frequency Limits

$$k_{zm} D \to m\pi \qquad \text{for} \qquad \omega \to \infty\,,$$

The Pekeris Waveguide Field

Spectral Regimes

- $0 < k_r < k_2$: The *continuous spectrum* where waves are radiating into the bottom, thus leaking energy away from the waveguide.
- $k_2 < k_r < k_1$: The *discrete spectrum* where the field is propagating vertically in the water and is exponentially decaying in the bottom. This part of the spectrum contains the discrete poles corresponding to lossless modes.
- $k_1 < k_r$: The *evanescent spectrum* where wave components in both water and bottom are exponentially decaying in the vertical.

[see Jensen, Fig 2.29]

Attenuation

Plane Harmonic Wave

$$\psi(x,t) = A\, e^{-i(\omega t - kx)}\,,$$

Linear Attenuation

$$\begin{aligned}\psi(x,t) &= A\, e^{-i(\omega t - kx) - \beta x}\,, \qquad \beta > 0 \\ &= A\, e^{-i[\omega t - k(1+i\delta)x]}\end{aligned}$$

Complex Wavenumber

$$\tilde{k} = k(1 + i\delta)\,.$$

Attenuation in dB/λ

$$\alpha = -20\log\left|\frac{\psi(x+\lambda,t)}{\psi(x,t)}\right| = -20\log\left[e^{-\delta k\lambda}\right] = 40\pi\,\delta\log e \simeq 54.58\,\delta\,.$$

Reciprocity

Transmission Loss Helmholtz Equation

$$\rho\,\nabla\cdot\left[\frac{1}{\rho}\,\nabla P(\mathbf{r},\mathbf{r}_s)\right]+k^2P(\mathbf{r},\mathbf{r}_s)=-4\pi\,\delta(\mathbf{r}-\mathbf{r}_s)\;.$$

Transmission Loss Reciprocity

$$\rho(\mathbf{r}_s)\,P(\mathbf{r},\mathbf{r}_s)=\rho(\mathbf{r})\,P(\mathbf{r}_s,\mathbf{r})\;.$$

[see Jensen, Fig 2.30]