Compressible Fluid Dynamics: Lect. 1

Admin: office hours? Reschedule 2 classes

HW, announcements, etc.

Textbooks (note: syllabus is evolving)

Grading 60% HW, class participation
40% term project
- written report + presentation
- will point out suggested topics during lecture
- e.g. review "classic" paper and expand (extra calculations, apply to specific problem...)

Compressible Flows (A.K.A. Gas Dynamics)

A flow may be considered *<u>incompressible</u> if a material element conserves volume as it is convected by the flow (show multi-media movie)

More intuitively, a flow is <u>compressible</u> if changes in fluid density play an essential role. (show multi-media movie, schlieren photography: see changes in index of refraction due to changes in density. SHOCK: discontinuity in density, velocity, etc.)

Compressibility may be important in many circumstances, most commonly, when typical velocities in the fluid exceed the speed of sound i.e.

$$\text{Mach \#} = \frac{v}{c} \gtrsim 1$$

we will see that this is a state dependent property and ∴ not constant

(If anyone knows speed of sound in air, $c \approx 350$ m/s or water, $c \approx 1500$ m/s, calc. v here.)

fluid which is
specific heats

oulli equation

(5.14)

hat compres-
Iach number
$\Delta\rho \approx c^2$; with

Let us now
npressibility.
uation (5.4),

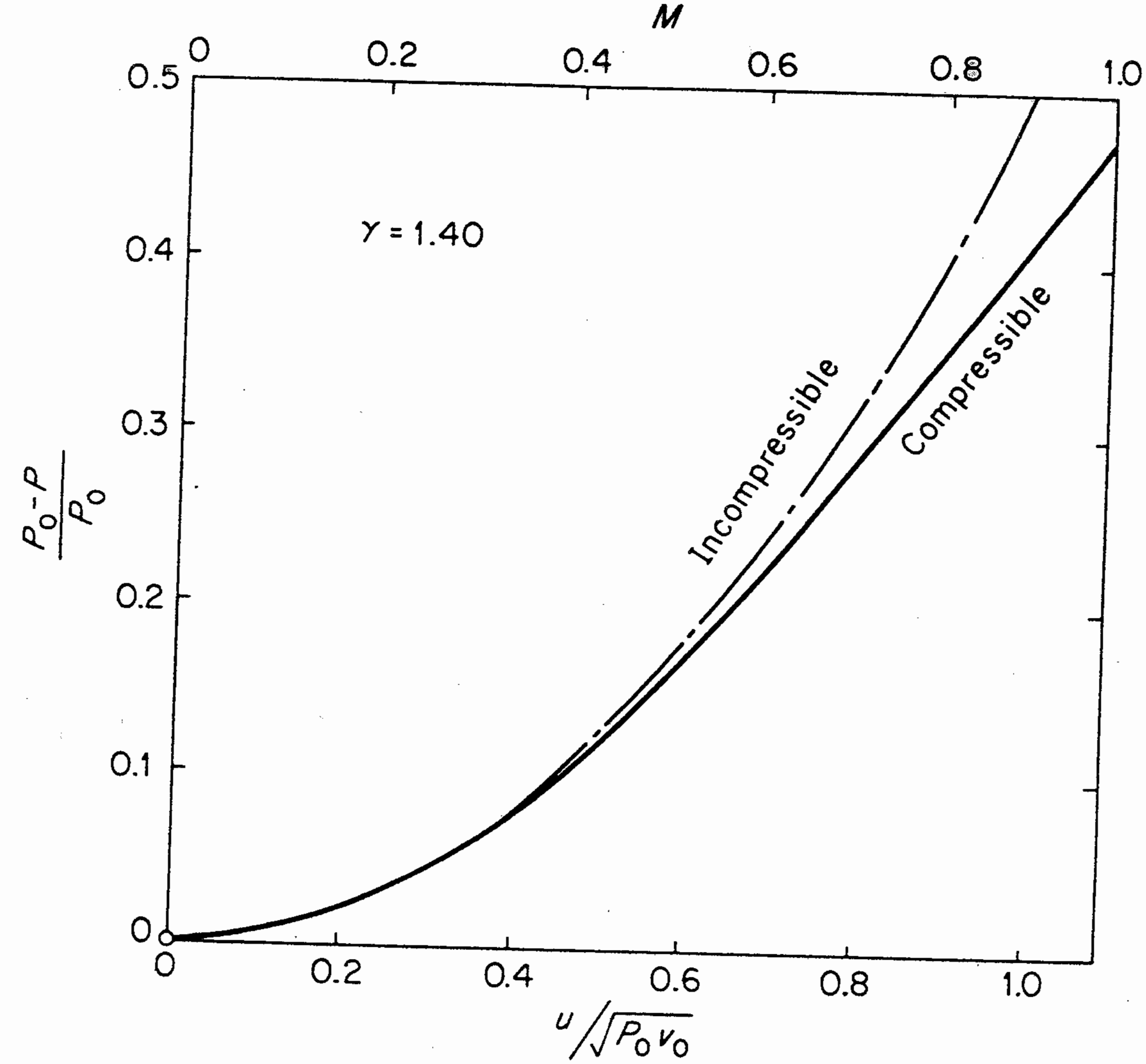


Figure 5.9
Pressure drop vs. velocity for a perfect gas with $\gamma = 1.40$.

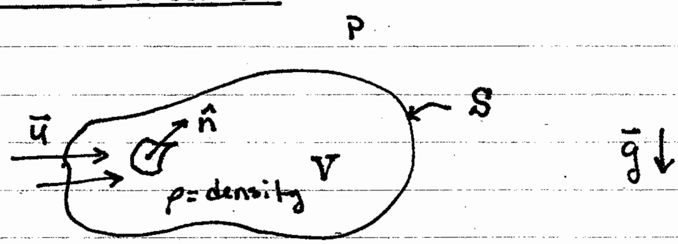
Review of Fluid Mech (chpt. 1 Thomson)

To find fundamental conservation laws, use Reynold's Transport Theorem:

$$\begin{pmatrix}\text{rate of change}\\ \text{of "stuff" in V}\end{pmatrix} = \begin{pmatrix}\text{rate of stuff}\\ \text{being created}\\ \text{in V}\end{pmatrix} + \begin{pmatrix}\text{stuff flowing}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{stuff flowing}\\ \text{out}\end{pmatrix}$$

"stuff" = mass, momentum, energy, entropy ~~and~~ etc.

Mass Conservation



$$\frac{d}{dt}\int_V \rho\, dV + \int_S \rho\, \bar{u}\cdot\hat{n}\, dS = 0$$ suppose V is fixed

$\downarrow$ diverdence thm. $\rho = \rho(\bar{x}, t)$

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \nabla\cdot(\rho\bar{u})\, dV = 0$$

Since V is arbitrary: $\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{u}) = 0}$

$$\underline{\frac{\partial \rho}{\partial t}} + \rho\nabla\cdot\bar{u} + \underline{u\cdot\nabla\rho} = 0$$

Recall $\frac{\partial}{\partial t} + u\cdot\nabla = \frac{D}{Dt}$ = material derivative

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{u} = 0}$$

(Recall for incompressible flows, $\frac{D\rho}{Dt} \approx 0 \Rightarrow \nabla\cdot\bar{u} = 0$)

<u>Conservation of momentum</u> (assume inviscid for now)

$$\frac{d}{dt}\int_V \rho \bar{u}\, dV + \int_S \rho \bar{u}\, \bar{u}\cdot\hat{n}\, dS = \underset{\text{(Forces!)}}{\text{momentum created}}$$

$$= \int_V \rho g\, dV + \int_S -P\hat{n}\, dS$$

$$\int_V \frac{\partial}{\partial t}(\rho\bar{u})\, dV + \int_V \nabla\cdot\underbrace{(\rho\bar{u}\bar{u})}_{\text{tensor}}\, dV = \int_V \rho\bar{g}\, dV - \int_V \nabla P\, dV$$

$$= \frac{\partial}{\partial x_j}(\rho u_j u_i)$$

$$\frac{\partial}{\partial t}(\rho\bar{u}) + \underbrace{\rho\bar{u}\,\nabla\cdot\bar{u}} + \bar{u}\cdot\nabla(\rho\bar{u}) = \rho\bar{g} - \nabla P$$

$$\frac{D}{Dt}(\rho\bar{u}) + \rho\bar{u}\,\nabla\cdot\bar{u} = \rho\bar{g} - \nabla P$$

$$\rho\frac{D\bar{u}}{Dt} + \bar{u}\Big(\overbrace{\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{u}}^{0\ \text{(cons. of mass.)}}\Big) = \rho\bar{g} - \nabla P$$

$$\boxed{\rho\frac{D\bar{u}}{Dt} = -\nabla P + \rho g}$$ Euler Equation

Note one important difference between compressible and incompressible flows already:

<u>Incompressible</u> Euler (3 eq) unknowns: $\bar{u}$, p (4) ✓
$\nabla\cdot\bar{u} = 0$ (1 eq)

<u>Compressible</u> Euler (3 eq) unknowns: $\bar{u}$, p, ρ !! (5!)
$\frac{D\rho}{Dt} + \rho\nabla\cdot\bar{u} = 0$ (1 eq)

Brief review of essential thermodynamics (Chapter 2)

A local thermodynamic state is fixed by any two thermodynamic variables. (e.g. p and s; or ρ and T, etc) (Equilibrium statement! Never true for transport! But we fudge this.)

⇒ Term Project paper:

Coleman and Mizel, "Existence of Caloric equations of state in thermodynamics" J. Chem Phys. vol. 40 (1964) 1116-1125.

Lighthill "Viscosity effects in waves of finite amplitude" in Batchelor and Davies Surveys in Mechanics, Cambridge University Press (on noslip cond.)

internal energy = $e = e(v,s)$

enthalpy = $h = h(s,p) = e + pv$

First law: $de = dq + dw = T\,ds - p\,dv$

reversible = no entropy produced

$\therefore\ dh = de + p\,dv + v\,dp = T\,ds + v\,dp$

Heat capacities: $C_v = \left.\frac{\partial e}{\partial T}\right|_v$ $\qquad \gamma = C_p/C_v$

$C_p = \left.\frac{\partial h}{\partial T}\right|_p$

Maxwell's relations (pg. 61 in Thomson

RTT for energy + entropy: $\rho \frac{D}{Dt}\left(e + \frac{u^2}{2}\right) = \text{viscous diss.} + \rho g \cdot u - \nabla \cdot \bar{q}$

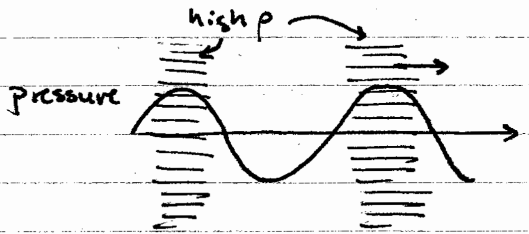
heat fl

$\rho \frac{Ds}{Dt} + \nabla \cdot (\bar{q}/T) \geq 0$

Clearly we need one more equation. Typically this final equation is cons. of energy (or possibly entropy).
→ Thermodynamics!

<u>Acoustics</u> - fluid motions associated w. the propagation of sound.

Start w. fluid medium at rest. $\bar{u}_0 = 0$, ρ_0, P_0
sound waves are <u>small</u> amplitude pressure fluctuations in the media. (how small is small? Look up #'s)



Neglect viscous dissipation; neglect heat transfer
⇒ the flow is ≈ <u>isentropic</u> $(s = s_0)$

Recall, for a pure substance, we only need two thermodynamic variables to fix the state of the system. e.g. P and s. ∴ $\rho(P,s)$ ①

Perturb about rest state: $\bar{u} = 0 + \bar{u}'$
$\rho = \rho_0 + \rho'$ $\quad P = P_0 + P'$

<u>Mass</u>: $\dfrac{\partial \rho}{\partial t} + \rho \nabla\cdot\bar{u} + \bar{u}\cdot\nabla\rho = 0$

$$\frac{\partial \rho'}{\partial t} + (\rho_0 + \cancelto{\text{small}}{\rho'})\nabla\cdot\bar{u}' + \cancelto{\text{small}}{\bar{u}'\cdot\nabla\rho'} = 0$$

② $\dfrac{\partial \rho}{\partial t} + \rho_0 \nabla\cdot\bar{u} = 0$ (dropping primes)

<u>Euler</u>: $(\rho_0 + \cancel{\rho'})\dfrac{D\bar{u}'}{Dt} = -\nabla P' + \cancelto{\text{neglect}}{\rho g}$

$\rho_0 \dfrac{D\bar{u}}{Dt} = -\nabla P$ (dropping primes)

$\rho_0\left(\dfrac{\partial \bar{u}}{\partial t} + \cancel{\bar{u}\cdot\nabla u}\right) = -\nabla P$

both <u>linear</u>

From 1 $$\frac{\partial \rho}{\partial t} = \underbrace{\left(\frac{\partial \rho}{\partial P}\right)_s}\frac{\partial P}{\partial t} + \left(\frac{\partial \rho}{\partial s}\right)_P \cancelto{0}{\frac{\partial s}{\partial t}}$$

$$\frac{1}{c_0^2} \equiv \left(\frac{\partial \rho}{\partial P}\right)_s$$

From 2 $$\frac{\partial}{\partial t}\left(\frac{\partial P}{\partial t} = -\rho_0 c_0^2 \frac{\partial u}{\partial x}\right)$$ (consider 1D)

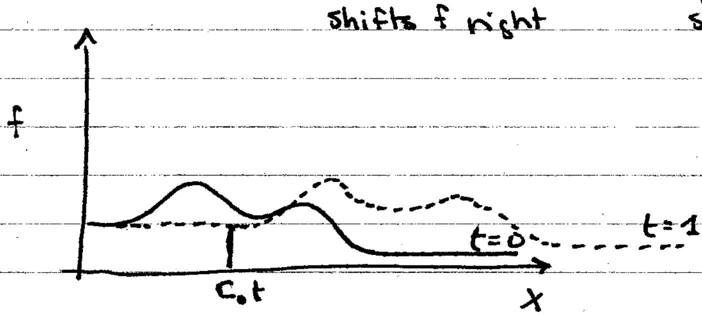
3 $$\frac{\partial}{\partial x}\left(\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x}\right)$$

$$\boxed{\frac{\partial^2 P}{\partial t^2} = \underset{\text{constant}}{\underset{\uparrow}{c_0^2}} \frac{\partial^2 P}{\partial x^2}}$$

Classical Wave Equation
(similarly for ρ and u, etc)

Solutions to the wave equation:

$$\boxed{P = \underset{\text{shifts } f \text{ right}}{\underset{\uparrow}{f(x - c_0 t)}} + \underset{\text{shifts } g \text{ left}}{\underset{\uparrow}{g(x + c_0 t)}}}$$



f and g can be any reasonable shape ~~given~~ fixed by the initial condition

Show that this is a solution: (let g = 0) let $\xi = x - c_0 t$

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial \xi}\frac{\partial \xi}{\partial t} = -c_0 \frac{\partial P}{\partial \xi} \equiv -c_0 P'$$

$$\frac{\partial P}{\partial x} = P'$$

plug into wave eq.

$$c_0^2 P'' = c_0^2 P'' \quad \checkmark$$

Note: we can interpret c_0 as the <u>speed of the wave</u>

All variables:
P, ρ, etc. are constant for an observer traveling @ $\frac{dx}{dt} = c_0$

$\Rightarrow$ the speed of sound in any fluid is $$\boxed{c^2 = \left(\frac{\partial P}{\partial \rho}\right)_s}$$

H.W. show $\vec{u}$ and ρ ~~[illegible]~~ are also governed by the wave. eq.

Speed of sound in an ideal gas:

$$Pv = RT \quad \Rightarrow \quad P = \rho R T$$

R ↑ specific gas const

For an isentropic process: $Pv^{\gamma} = \text{const} \Rightarrow P\rho^{-\gamma} = \text{const}$

where $\gamma = c_p / c_v$ (ratio of specific heats)

$$c^2 = \left(\frac{\partial P}{\partial \rho}\right)_s = \gamma \,\text{const.}\, \rho^{\gamma-1} = \gamma \rho^{\gamma-1} P \rho^{-\gamma} = \frac{\gamma P}{\rho}$$

$$= \gamma R T$$

$$\boxed{c_{\substack{\text{ideal}\\ \text{gas}}} = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\gamma R T}}$$

$\gamma = 1.4 \qquad P = 1\,\text{atm} \approx 10^5\ \text{N/m}^3 \qquad \rho \approx 1.25\ \text{kg/m}^3$

$\Rightarrow \quad c_{air} = 335\ \text{m/sec}$

Speed of sound in a liquid

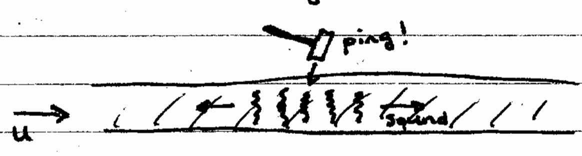
Define in terms of $k_s = \rho \left(\frac{\partial P}{\partial \rho}\right)_s$ = isentropic bulk modulus

$$\boxed{c^2 = \frac{k_s}{\rho}}$$

$c_{water} \approx 1500\ \text{m/s}$

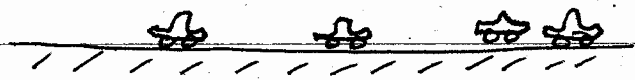
Sound waves in a moving medium:

E.g.

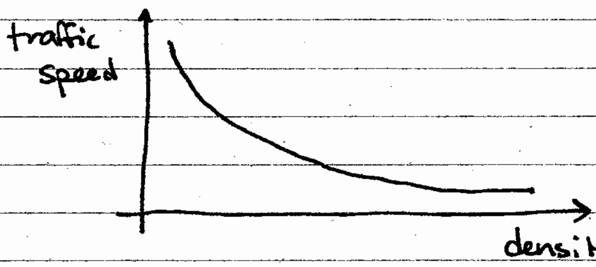


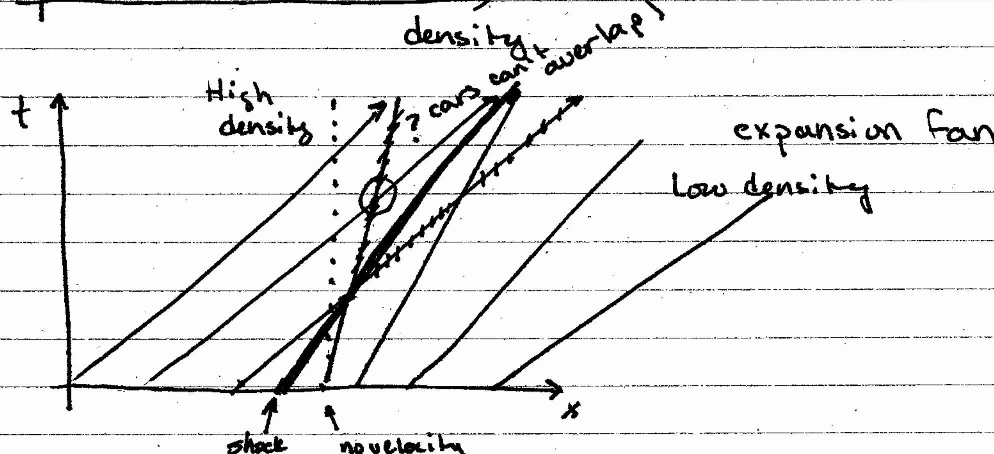
$V_{pulse} = u \pm c$

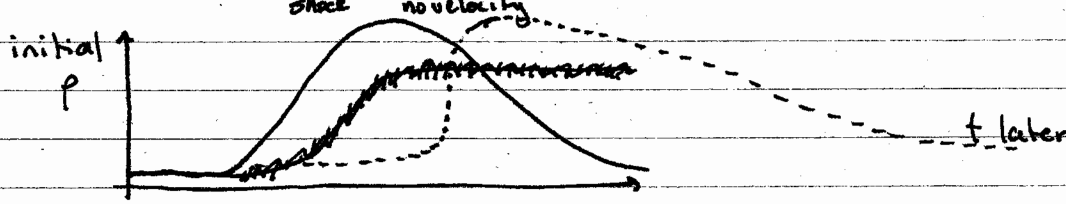
Traffic analogy



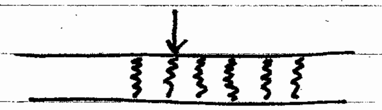
ρ = "density" = # cars/unit length





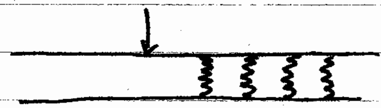


$M < 1$ subsonic
($M < 0.3 \approx$ incompressible)
$M \approx 1$ transonic



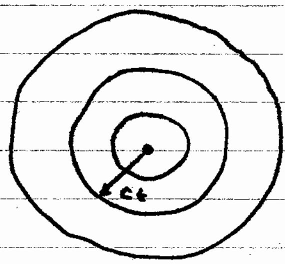
both upstream and downstream flow "see" the disturbance

$M > 1$ supersonic

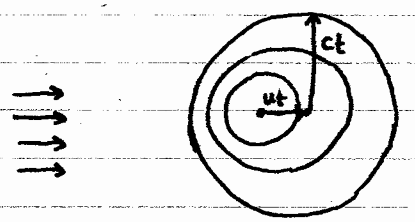


upstream "sees" nothing!! Pulse cannot affect upstream conditions.

In 3D



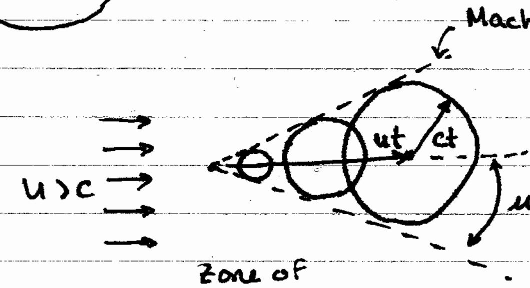
stationary fluid



u
(= object moving to left @ velocity u)

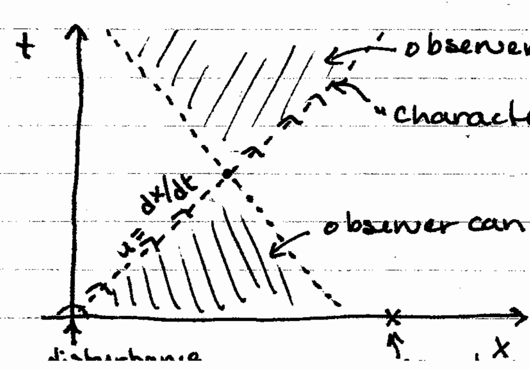
$u < c$

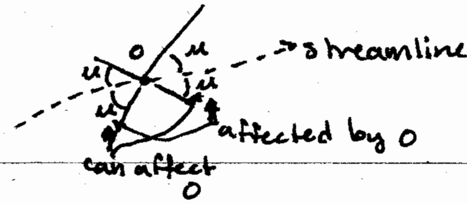
For a stationary observer, ~~?~~ frequency is decreased due to Doppler effect.)



$$\sin \mu = \frac{c}{u} = \frac{1}{M}$$

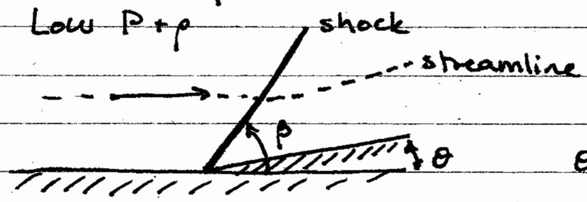
Wave equations and causality (Do this earlier w. wave eq.)



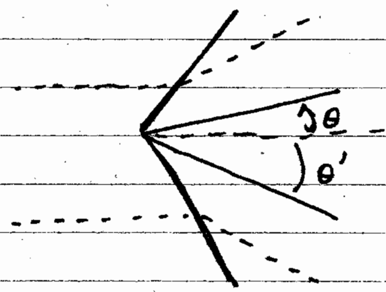


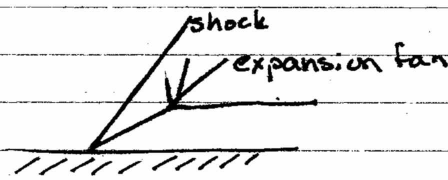
The Shock is a discontinuity in physical ~~properties~~ quantities ($p, \rho, \bar{u}$, etc.)

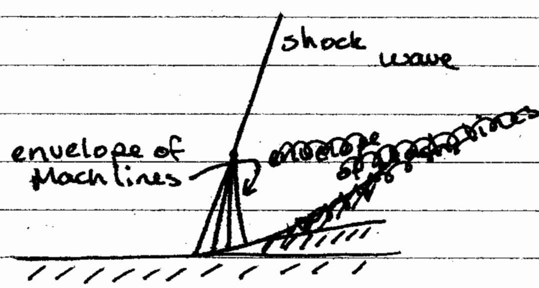
Supersonic flow past an object:

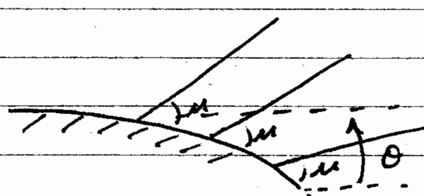


$\theta \to 0$, $\beta \to \mu$ (but shock becomes weaker)
Mach waves









expansion Mach wave