Coordinate Systems and Separation of Variables

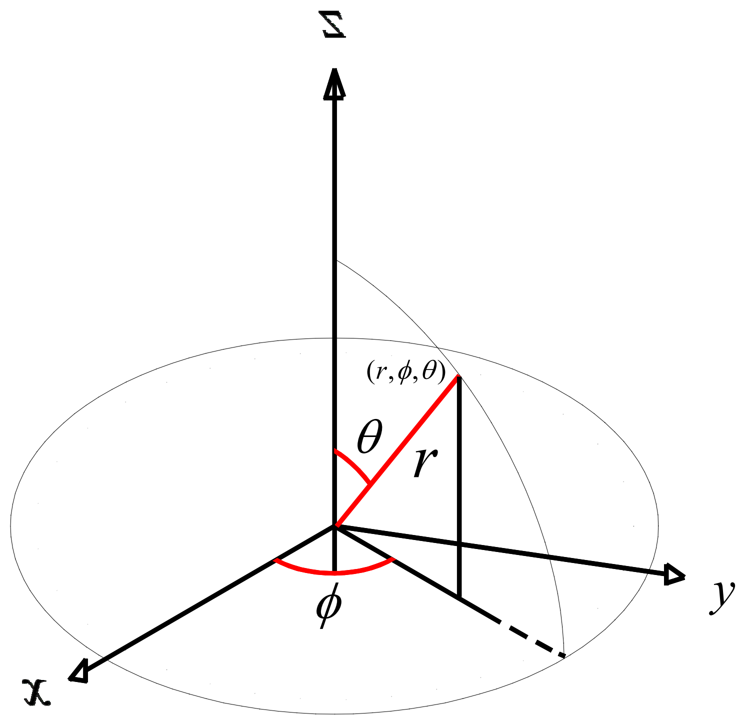
Revisiting the wave equation… $$\nabla^2\psi + \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

where previously in Cartesian coordinates, the Laplacian was given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We are now faced with a spherical polar coordinate system, with the motivation that we might employ the *separation of variables* technique to solve the wave equation where spherical symmetries are involved.

Spherical Polar Coordinates



$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left[\sqrt{x^2 + y^2}/z\right]$$

$$\phi = \tan^{-1}\left[y/x\right]$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Separation of Variables

Objective is to restate problems in alternative orthogonal coordinate systems such that for the particular boundary conditions in force, the solutions can be assumed to separate such that…

$$\psi(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$$

Azimuth dependence: $$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

Elevation dependence: $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

Radial dependence: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{n(n+1)}{r^2}R = 0$$

Time dependence: $$\frac{1}{c^2}\frac{d^2T}{dt^2} + k^2T = 0$$

Elevation Dependence: Legendre Functions

Legendre polynomials: $P_n^0(x) = P_n(x)$

Represent fields uniform in azimuth coordinate ϕ

Legendre polynomials with integer order $m \neq 0$ represent other kinds of fields with harmonic variations in azimuth.

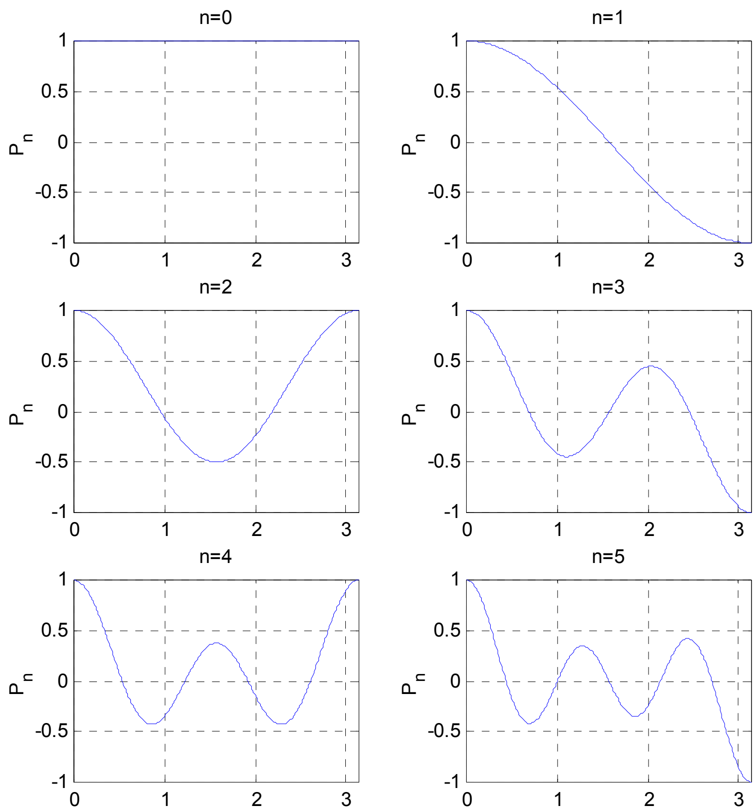
The Associated Legendre polynomials

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

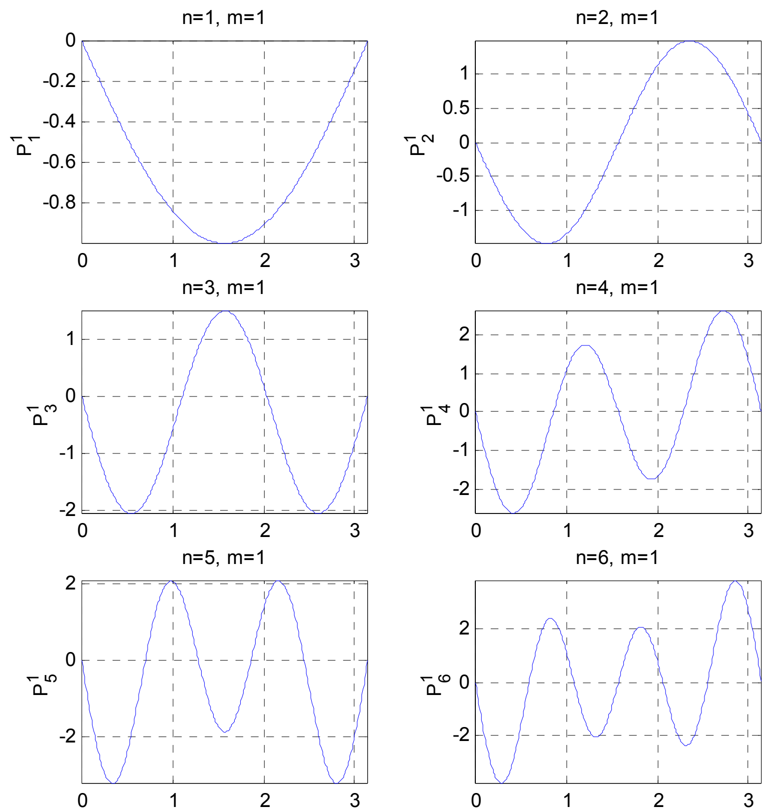
Legendre functions available in MATLAB via call to **legendre**(N,X).

Selected Legendre Functions

$$P_n^0(x) = P_n(x)$$



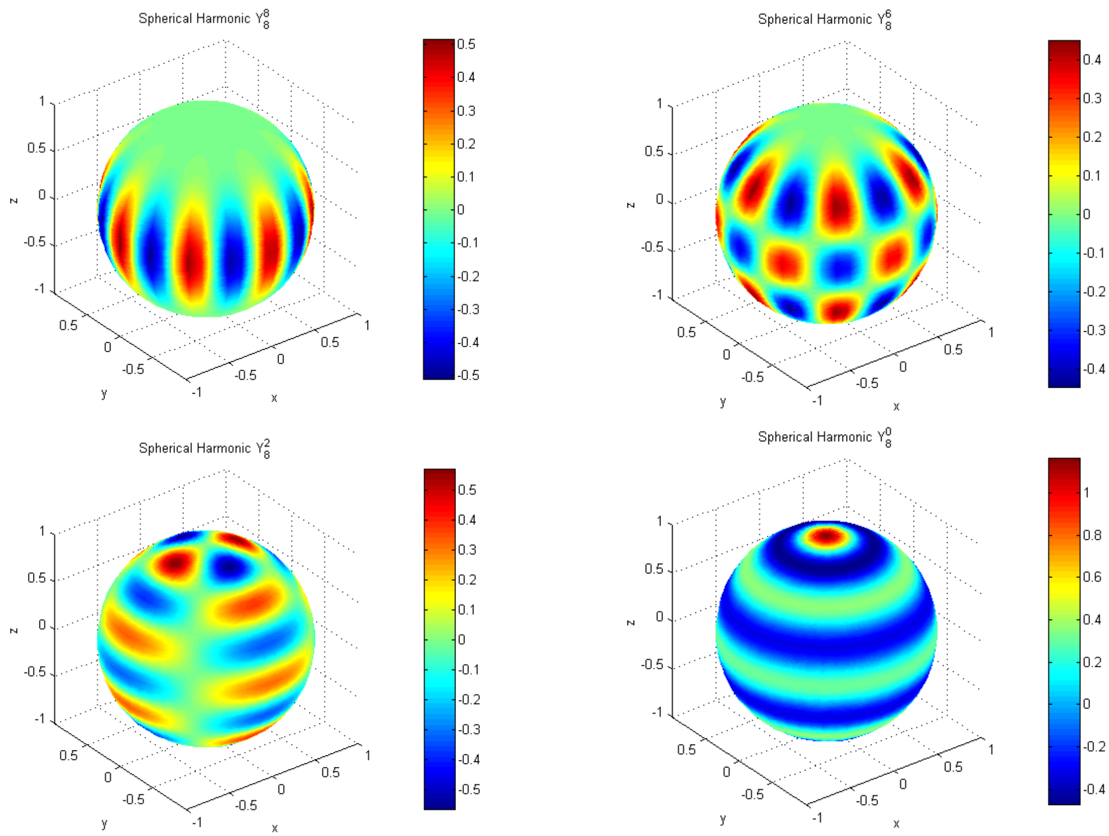
$$P_n^m(x) \text{ for } m = 1$$



Spherical Harmonics

Lump together azmuthal and elevation dependence to arrive at spherical harmonics…

$$Y_n^m(\theta,\phi) \equiv \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)\exp im\phi$$



Radial Dependence - Spherical Bessel Functions

Solution to a differential equation that is closely related to the ordinary Bessel equation...

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+k^2-\frac{n(n+1)}{r^2}\right]R_n(r)=0$$

Substituting $R_n(r)=\frac{1}{r^{1/2}}u_n(r)$ gives

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+k^2-\frac{(n+1/2)^2}{r^2}\right]u_n(r)=0$$

Standing wave solutions

$$j_n(x)\equiv\left(\frac{\pi}{2x}\right)^{1/2}J_{n+1/2}(x)$$

$$y_n(x)\equiv\left(\frac{\pi}{2x}\right)^{1/2}Y_{n+1/2}(x)$$

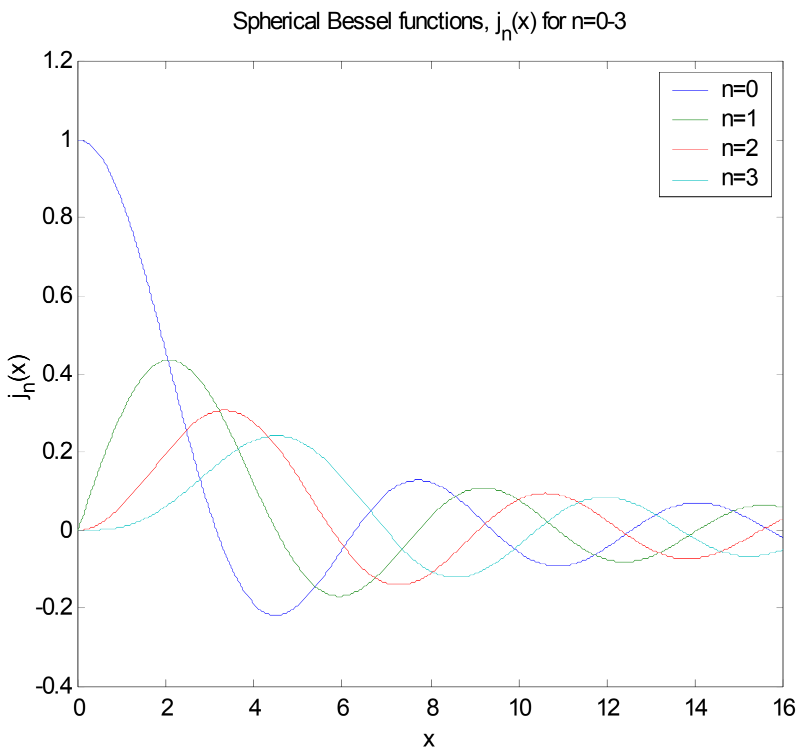
OR

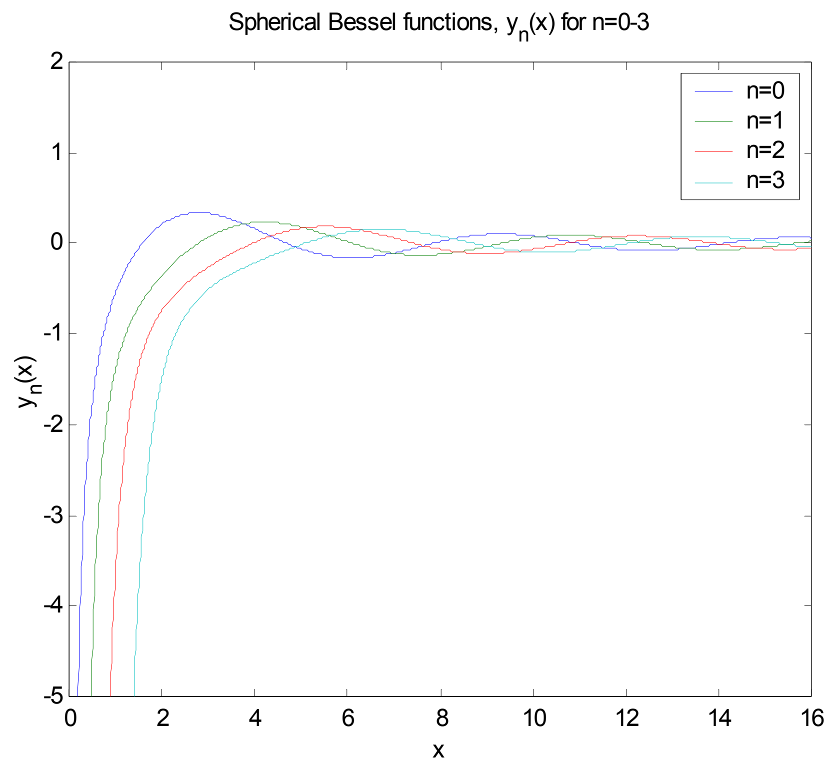
Traveling wave solutions

$$h_n^{(1)}(x)\equiv j_n(x)+iy_n(x)=\left(\frac{\pi}{2x}\right)^{1/2}\left[J_{n+1/2}(x)+iY_{n+1/2}(x)\right]$$

$$h_n^{(2)}(x)\equiv j_n(x)-iy_n(x)=\left(\frac{\pi}{2x}\right)^{1/2}\left[J_{n+1/2}(x)-iY_{n+1/2}(x)\right]$$

Selected Spherical Bessel Functions





Asymptotic forms for Bessel functions

In odd numbers of dimensions, we are able to express Bessel functions exactly via trigonometric expansions.

Correspondence between planar and cylindrical expansions

$$P(k_x, k_y, z_0) \leftrightarrow P_n(r, k_z)$$

$$e^{ik_z(z-z_0)} \leftrightarrow \frac{H_n^{(1)}(k_r r)}{H_n^{(1)}(k_r a)}$$

Important Connections

- Infinite domains → Continuous transforms
- Domain periodicity → Fourier series

- Equivalence of coordinate representations given the right integrations
- Basis functions result from solutions to differential equations
- Solutions to the Bessel differential equation $$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left(k_r^2-\frac{n^2}{r^2}\right)R=0$$
- Are given by $J_n(k_r r)$ and $Y_n(k_r r)$

- These are Bessel functions of the first and second kind (the latter also referred to as Neumann functions). As the domain is periodic (at least in this dimension), n must be an integer.

Interior and Exterior problems

These functions are analogous to the plane wave exponentials, and in fact, have the asymptotic forms

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - n\pi/2 - \pi/4)$$

$$Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin(x - n\pi/2 - \pi/4)$$

Solution to interior problem:

$$p(r,\phi,z,\omega) = \sum_{n=-\infty}^{\infty} e^{in\phi} \frac{1}{2\pi} \int_{-\infty}^{\infty} C_n(k_z,\omega) e^{ik_z z} J_n(k_r r)\, dk_z$$

Need one measurement surface for each unknown coefficient function.

Review

- Plane wave expansions etc
- Plane wave solutions useful in Cartesian geometries.

Motivation

- To treat propagation and scattering problems involving spherical geometries and symmetries.

Real-Parts of Selected Spherical Harmonics