Question 1:

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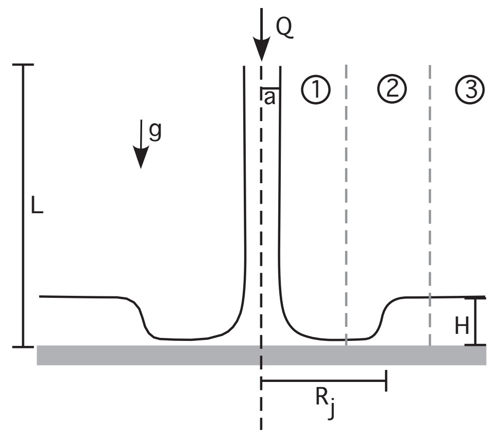


When a jet of fluid impinges on a horizontal plate, the fluid flows radially outward away from the jet in a thin film. At a distance R_j from the jet, the thickness of the film suddenly increases. This phenomenon is known as a hydraulic jump. In general, the flow in the jump region is turbulent.[1] Consider a jet with volume flux, Q, and radius, a. The fluid in the jet can be approximated as **inviscid** with density, ρ. The acceleration of gravity is $-g$. You may assume the flow is axially symmetric. You may also assume $a \ll H \ll R_j$.

The hydraulic jump can be divided into three regions as indicated in the figure: (1) the upstream region, $a < r < R_j$, (2) the jump region $r \sim R_j$ and (3) the downstream region $r > R_j$. Follow the steps below to find an expression for the steady state jump radius, R_j.

a.) Using dimensional analysis, find a complete set of independent Pi groups for this system.

Solution: The jump radius could depend on:

$$R_j = \Phi(\rho, Q, a, H, g)$$

However, the only parameter in this list that includes mass is ρ so there is no way to eliminate $[M]$ if ρ in included in a Pi group. Hence

$$R_j = \Phi(Q, a, H, g)$$

Whether we include H as a parameter depends on the far-field boundary condition. In many hydraulic jump experiments, H can be tuned independently

[1]This turbulence may be suppressed in sufficiently viscous fluids but here we will only consider the turbulent case.

by placing a wall of height H at a radius much larger than the jump radius. However H may also be determined by a balance of gravity and surface tension. Since the far-field conditions were not specified, either answer is acceptable. L may also be included as a possible paramter. μ should not be included as the flow is modeled as inviscid. The number of Pi groups $= n - k$ (using the list above) where $n = 5$ and $k = 2$ ($[L]$ and $[T]$). Thus

$$\Pi_1 = \frac{R_j}{a}, \qquad \Pi_2 = \frac{H}{a}, \qquad \Pi_3 = \frac{a^5 g}{Q^2}$$

Note that $\Pi_3 = a^5 g/Q^2 \sim ga/v^2 = 1/Fr_a$ where Fr_a is the Froude number defined using a as a characteristic length scale.

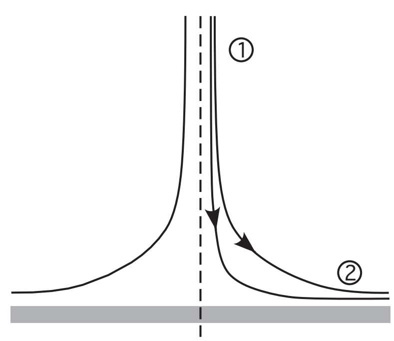
b.) In the downstream region (3), the height of the free surface is a known constant, H (i.e., the height of the free surface is NOT a function of r). Find an expression for the average velocity in the downstream region.

Solution: By conservation of mass:

$$Q = 2\pi r H v_r(r) \Rightarrow v_r(r) = \frac{Q}{2\pi H r}$$

c.) The upstream region (1) consists of a jet impinging on a horizontal plate. Derive an expression for the height of the free surface in this region, $h(r)$, and the average radial velocity. Recall that you may assume $a \ll R_j$.

Solution:



By conservation of mass:

$$Q = v_1 \pi a^2 = v_2 2\pi r h(r)$$

Following a streamline from point 1 to point 2 along the free surface, we can apply Bernoulli (note that $p = p_a$ everywhere along the streamline).

$$p_a + \frac{1}{2}\rho v_1^2 + \rho g L = p_a + \frac{1}{2}\rho v_2^2$$

$$\Rightarrow v_2 = \sqrt{v_1^2 + 2gL} = \sqrt{Q^2/(\pi a^2)^2 + 2gL} = \text{constant} \equiv V$$

The Froude number in terms of L, the distance of the jet source above the plate, Fr_L is given by $v^2/gL \sim Q^2/a^4 gL$. In the limit $Fr_L \gg 1$,

$$V \approx \frac{Q}{\pi a^2}$$

(i.e. $v_1 \approx v_2$). Combining this with conservation of mass:

$$h(r) = \frac{Q}{2\pi r V} = \frac{a^2}{2r}$$

d.) Now that you have obtained an expression for the height of the free surface and the velocity on either side of the jump, derive an expression for R_j in terms of known parameters.

Solution:

	$r < R_j$	$r > R_j$
$v(r)$	$Q/\pi a^2$	$Q/2\pi H r$
$h(r)$	$a^2/2r$	H

Drawing a control volume annulus with inner radius at $R_j - \epsilon$ and outer radius $R_j + \epsilon$ we can write down an expression for conservation of momentum[2]. (For simplicity will take $p_a = 0$.

$$\frac{d}{dt}\int_{cv} \rho \mathbf{v}\, dV + \int_{cs} \rho \mathbf{v}(\mathbf{v} - \mathbf{v}_c)\cdot \mathbf{n}\, dA = -\int_{cs} p_a \mathbf{n}\, dA - \int_{cv} \rho g \mathbf{e}_z\, dV$$

The first term is zero since the system is in steady state. The pressure term is also zero as the pressure at the surface is zero and, since the streamlines are roughly straight entering and exiting the control volume, the pressure is ≈ 0 in the flow. We can also rewrite the last term using the Divergence Theorem:

$$\int_{cs} \rho \mathbf{v}(\mathbf{v} - \mathbf{v}_c)\cdot \mathbf{n}\, dA = -\int_{cs} \rho g z \mathbf{n}\, dA$$

In the $\mathbf{e}_r$ direction:

$$\rho \left(\frac{Q}{2\pi H (R_j + \epsilon)}\right)^2 2\pi(R_j + \epsilon) H - \rho V^2 2\pi (R_j - \epsilon) h|_{(R_j - \epsilon)}$$
$$= -\rho g \left[\frac{1}{2} H^2 2\pi(R_j + \epsilon) - \frac{1}{2} h^2|_{(R_j - \epsilon)} 2\pi (R_j - \epsilon)\right]$$

[2] Note that we CANNOT match the velocities at the jump! Since the upstream and downstream heights are different, by conservation of mass, the upstream and downstream velocities must be different.

Take the limit as $\epsilon \to 0$:

$$\rho\left(\frac{Q}{2\pi H R_j}\right)^2 2\pi R_j H - \rho\left(\frac{Q}{\pi a^2}\right)^2 2\pi R_j \frac{a^2}{2R_j} = -\rho g\left[H^2\pi R_j - \left(\frac{a^2}{2R_j}\right)^2 \pi R_j\right]$$

Define $Fr_H = \frac{Q^2}{ga^4 H}$, $R = R_j/a$, and $A = a/H$. After some algebra, the equation above reduces to a quadratic in R and we can write the solution as:

$$R = \frac{AFr_H}{2\pi^2}\left[1 \pm \left[1 - \left(\frac{\pi^2}{Fr_H}\right)^2\left(\frac{2Fr_H}{\pi} - 1\right)\right]^{1/2}\right]$$

If $Fr_L \gg 1$ and $L/H \gg 1$ then $Fr_H \gg 1$. Hence

$$R \approx \frac{AFr_H}{2\pi^2}\left[1 \pm \left(1 - \frac{\pi^2}{Fr_H}\right)^{1/2}\right]$$

Thus

$$R \approx \frac{A}{2} \qquad \text{or} \qquad R \approx \frac{AFr_H}{2\pi^2}$$

Or in dimensional form:

$$R_j \approx \frac{a^2}{2H} \qquad \text{or} \qquad R_j \approx \frac{Q^2}{2\pi^2 g a^2 H^2}$$

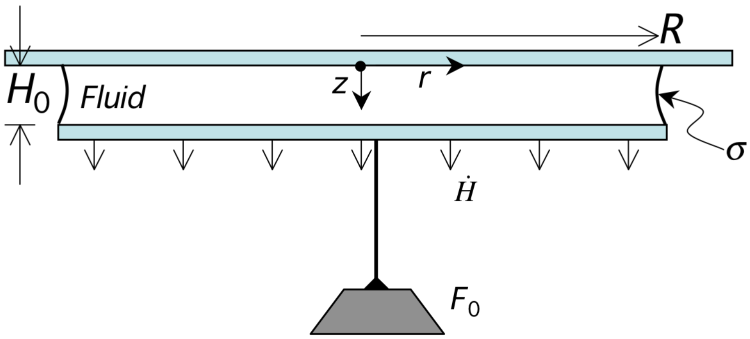
To select the relevant solution, note that the first one implies that there is no jump. I.e. using the first solution, $h(R_j) = \frac{a^2}{2a^2/(2H)} = H$ hence there is no jump (the transition from the upstream to the downstream region is smooth). Thus, the jump solution is the second one.

e.) Convert your answer from part (d) into dimensionless form using your Pi groups from part (a).

$$\pi_1 = (2\pi^2\pi_2^2\pi_3)^{-1}$$

Q2. Adhesive Separation of Two Disks [20pts].

It is a matter of common experience that it can take a large force to separate two surfaces that are joined by a thin layer of a viscous fluid which acts as an adhesive. Some insects such as aphids are believed to exploit this fact by using thin adhesive-like liquid films on their feet to enable them to walk inverted across the ceiling. In this question we consider the model problem of a cylindrical disk (radius R) separated by a constant initial thickness H_0 ($\ll R$) with a thin cylindrical film of a viscous incompressible Newtonian fluid (of viscosity μ and density ρ) in the gap.



The plates are to be separated by a <u>constant force</u> F_0 that is imposed on the lower plate as shown in the figure opposite. We wish to solve for the separation profile $\dot{H}(t)$ as a function of time (where the overdot indicates a time derivative).

(a) The cylindrical fluid film shown has a free surface with a surface tension σ. Use dimensional analysis to provide a dimensionless constraint for conditions under which all capillary effects (i.e. interfacial force contributions) are negligible in the subsequent analysis.

If the additional force from surface tension is small compared to other forces in the problem then we immediately see from inspection that we require $\sigma R/F \ll 1$.

If you consider this from point of view of full dimensional analysis (NOT required) then we seek to find a functional relationship of the form:

$$\dot{H} = f(F_0, \mu, \rho, \sigma, D, H_0)$$

We thus find that we have $n = 7$, $k = 3$ and $n\text{-}k = 4$ dimensionless groups; choosing F (flow), μ (fluid, since we expect flow in thin gap to be viscously dominated) and R (geometry) we thus find that for most general case:

$$\frac{\dot{H}\mu R}{F_0} = \phi\left(\frac{\sigma R}{F_0}, \frac{H_0}{R}, \frac{\rho F_0}{\mu^2}\right)$$

the fourth group should be familiar from homework (see 7.18). If surface tension effects and inertial effects are completely negligible then this expression simplifies to the following form:

$$\dot{H}\mu R/F_0 = \phi_2\left(H_0/R\right) .$$

(b) Let us consider the first instant in time (denoted $t = 0^+$) after the constant force is applied to the disk when the sample is cylindrical as shown. Write down the appropriate simplified form of the Navier-Stokes equations (in cylindrical coordinates) together with the appropriate boundary conditions, and any dimensionless criteria which must be attained for this simplification to be valid.

As usual for a locally-fully-developed and quasi-steady flow, we scale radial lengths with R, axial (gap) scales with H_0 , velocity with $\dot{H}$ (which needs to be found as part of the problem) and time with τ. The radial component of the Navier-Stokes equation then simplifies to:

$$0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \text{ provided } H/R \ll 1,\ \left(\rho \dot{H} R/\mu\right)\left(H/R\right)^2 \ll 1, \text{ and } H^2/\nu\tau \ll 1$$

where the characteristic scale for pressure in a lubrication flow is thus expected to be $p_c \sim \mu\left(\dot{H}/H\right)\left(R^2/H^2\right)$ (i.e. a viscosity times a deformation rate times the $\left(R/H\right)^2$ factor). The appropriate boundary conditions on velocity are $v_r = 0$ at $z = 0, H$ and for the pressure we expect $p = p_{atm}$ at $r = R$.

(c) Find the resulting form of the radial velocity profile in terms of the (as yet unknown) radial pressure gradient.

Integrating the velocity field twice and applying boundary conditions we obtain

$$v_z(r,t) = \frac{H(t)^2}{2\mu}\left(-\frac{\partial p}{\partial r}\right)\left[\frac{z}{H(t)} - \frac{z^2}{H(t)^2}\right] \tag{2.1}$$

Note that the directionality of v_r is determined by the SIGN of the pressure gradient. If $\partial p/\partial r > 0$ then the flow will be INwards (and vice-versa).

(d) Use conservation of mass to write down a kinematic condition that relates the volumetric inflow through an annulus of radius $r < R$ and the axial displacement rate $\dot{H}(t)$ of the disk. Use this result plus appropriate boundary conditions to find an equation for the pressure profile across the sample. Is the pressure at the center of the disk greater than or less than atmospheric?

The Navier-Stokes equation in the axial direction yields simply $0 = -\frac{\partial p}{\partial z} + \rho g$ so there are only hydrostatic variations axially and the pressure is only a function of r. Applying a control volume to a cylindrical column of fluid of radius r and height $H(t)$ we obtain:

	$\mathbf{n}$	dA	$\boldsymbol{v}$
Top surface	$\mathbf{e}_z$	$2\pi r dr$	$v_z = \dot{H}\mathbf{e}_z$
Radial surface	$\mathbf{e}_r$	$2\pi r dz$	$v_r(z)\mathbf{e}_r$

Using these values we obtain:

$$\frac{dM_{cv}}{dt} + \pi r^2 \dot{H} + \int_0^H \left[v_z(r)\mathbf{e}_r\right]\cdot \mathbf{e}_r (2\pi r dz) = 0 \tag{2.2}$$

Substituting 2.1 into 2.2 and integrating we obtain:

$$r\dot{H} = -\frac{H^2}{\mu}\left(-\frac{\partial p}{\partial r}\right)\left[\frac{z}{2H^2} - \frac{z^3}{3H^2}\right]_0^H = \frac{H^3}{6\mu}\left(\frac{dp}{dr}\right) \tag{2.3}$$

rearranging we find $\int_{p(r)}^{p_{atm}} dp = \int_r^R \frac{6\mu\dot{H}}{H^3} r dr$ and integrating radially and applying the boundary conditions we finally obtain:

$$p(r) = p_0 - \frac{3\mu\dot{H}}{H^3}\left[R^2 - r^2\right] \tag{2.4}$$

The pressure thus varies quadratically across the gap. It is minimum at the middle with a maximum (gage) pressure drop of $\Delta p = p(r=0) - p_{atm} = -3\mu\dot{H}R^2/H^3$. If we compare this with our original scaling estimate we find we were off by a factor of –3 only! Note that if we change the SIGN of $\dot{H}$ so that we are squeezing the plates together then we get a large positive pressure resisting the squeezing.

(e) Integrating this result for the (gauge) pressure across the plate we find that the net viscous force exerted by the fluid on the lower plate (with outwards facing normal $\mathbf{n} = -\mathbf{e}_z$) acts axially upwards on the plate and is given by.

$$\mathbf{F}_v = \int_0^R -\left[p(r) - p_0\right]\mathbf{n} dA = \int_0^R -\left[p(r) - p_0\right](-\mathbf{e}_z)\, 2\pi r dr = -\frac{3\pi}{2}\mu R^4 \frac{\dot{H}(t)}{H(t)^3}\mathbf{e}_z \tag{2.5}$$

Note that this is consistent with our original expectations from dimensional analysis also, which indicated a functional form: $\mu R\dot{H}/F_0 = (-2/3\pi)\left(H_0/R\right)^3$.

(f) Assuming that this expression remains valid for all future times and plate separations, we can find an expression for the plate separation as a function of time. A force balance on the plate gives

$$F_0 - F_v = m_{plate}\frac{d^2H}{dt^2}$$

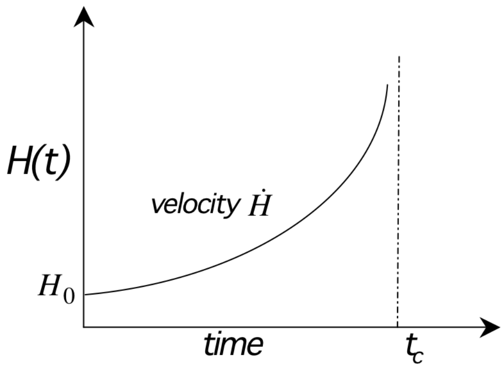
If we ignore the inertial mass of the plate and assume the forces are in quasi-steady equilibrium then we find that the motion of the plate is given by

$$F_0 = \frac{3\mu\pi R^4}{2H^3}\frac{dH}{dt} \quad \text{or} \quad \int_0^t \frac{2F_0}{3\mu\pi R^4}dt = \int_{H_0}^H \frac{dH}{H^3} \tag{2.6}$$

Integrating and evaluating we obtain

$$\frac{1}{H^2} = \frac{1}{H_0^2} - \frac{4F_0 t}{3\pi\mu R^4} \quad \text{or} \quad \frac{H}{H_0} = \frac{1}{\sqrt{1 - \frac{4F_0 H_0^2}{3\pi\mu R^4}t}} \tag{2.7}$$

The profile thus 'blows up' or shows a finite-time singularity at a critical time $t_c = 3\pi\mu R^4/4F_0H_0^2$. The equation (2.7) is sketched overleaf:



(g) The time-dependent solution obtained in (f) must fail at some point. Use dimensional analysis to provide an estimate for a characteristic time τ for this constant force separation in terms of the instantaneous position and velocity and thus provide a dimensionless constraint for when your solution in (f) is valid.

The instantaneous velocity is $\dot{H}$ and the instantaneous position is H, the characteristic time scale for the flow is thus $\tau \sim l_c / V_c = H / \dot{H}$. You can either evaluate this directly or note from equation (2.6) that $\dfrac{H}{\dot{H}} \equiv \tau = \dfrac{3\pi\mu R^4}{2F_0 H_0^2}$. For the flow to be quasi-steady flow we require $H^2 / \nu\tau \ll 1$; substituting for τ we thus obtain:

$$H^4 \ll \frac{3\pi\mu^2 R^4}{2\rho F_0} \quad \text{or} \quad \frac{H}{R} < \left(\frac{3\pi\mu^2}{2F_0\rho}\right)^{1/4}$$