Question 1:

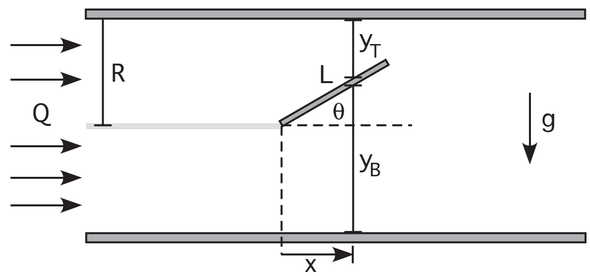


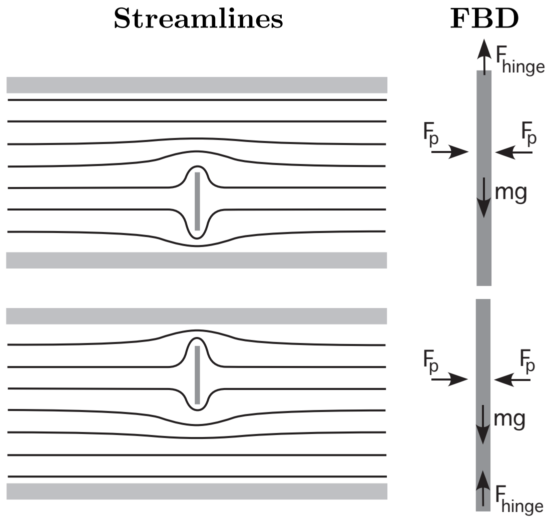
Figure 1: Schematic

Consider a channel of height $2R$ with rectangular cross-section as shown in the sketch. A hinged plank of a length $L < R$ and at an angle θ is located at the center of the channel. (You may assume that L is small compared to the length perpendicular to the sketch.) The plank has a mass per unit width, m. A constant volume flux per unit width, Q, of an inviscid, incompressible fluid is applied at the inlet of the channel.

Find all equilibrium values for θ (include BOTH stable and unstable equilibria!) To simplify your calculation, you may assume that we have chosen the mass, m, and volume flux, Q, such that for *at least one* equilibrium state, θ is small.

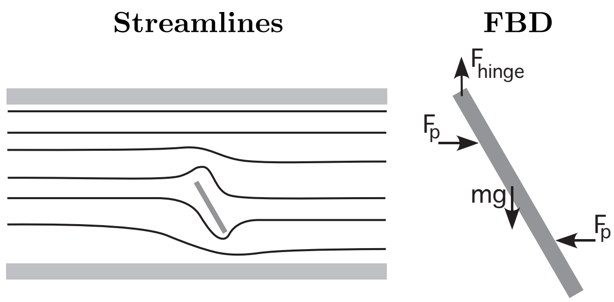
Solution:

(4 points total, 1 for each equilibrium and 1 each for stability.) Before we do any calculations we can see by inspection that for *inviscid* flow, there are two trivial equilibrium positions. If the plank is vertically aligned, the pressures on the front and back balance (since the flow is symmetric) and the weight of the plank is supported by the hinge:



Thus there is an equilibrium at $\boxed{\theta = \pi/2}$ and at $\boxed{\theta = 3\pi/2}$. Note, neither of these are physically realizable since, in a real fluid that has some viscosity, separation at the plank ends will break the front/back symmetry.

To find the stability of there two solutions, we perturb away from equilibrium and inspect the resulting torques. For the $\theta = 3\pi/2$ solution:



Considering the torque about the hinge, both gravity and pressure act to restore the plank to its equilibrium position so $\theta = 3\pi/2$ is $\boxed{\text{STABLE}}$.

Drawing a similar picture for $\pi/2$, we find that the pressure is stabilizing and gravity is destabilizing so we need to do a calculation to figure out which one wins (we will come back to this after the next section).

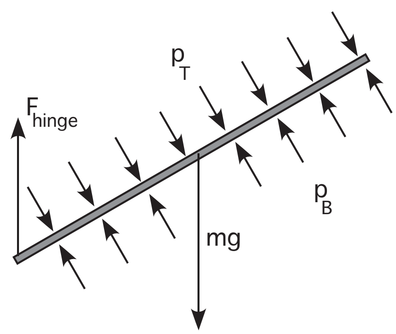
(15 points total: 3 points for mass conservation, 3 points for conservation of momentum (Bernoulli), 6 points for force and torque balance on the plate, 2 points to put it all together and find θ_{eq} (i.e. integrate!) and 1 point for stability.) Solving this problem exactly is nontrivial since technically, we would need to solve $\nabla^2\psi = 0$ with no flux boundary conditions at the plank and at the top and bottom channel walls. To approximate the equilibrium conditions, we need to estimate the torque on the plank. Since there are several reasonable assumptions you can make at this point, credit was given if you (1) made a reasonable set of assumptions (2) applied conservation of mass, conservation of momentum (in this case Bernoulli) correctly, and (3) applied force and torque balance on the plank in a way that is consistent with your reasonable assumptions.

Note that symmetry is broken in this problem because, if the plank is tilted $\theta \neq 0$, it is no longer centered in the channel. Even if you ignored this broken symmetry and assumed your plank is in an infinite flow, there is still a net torque on the plank due to the flow since the stagnation points on the front and back of the plank are not aligned (see the perturbation picture above) so the torque arising from pressure can counter-balance the torque due to gravity.

For small θ, one reasonable approximation to make is that roughly half of the flux goes above the plank and half below[1] . Thus, by conservation of mass,

$$\frac{Q}{2} = v_T(x)y_T(x) = v_B(x)y_B(x). \tag{1}$$

From the FBD diagram:



we can write down a vertical force balance:

$$mg - F_{hinge} = \int_0^L \left[p_B(x') - p_T(x')\right] \cos\theta \, dx' \tag{2}$$

and torque balance around the center of the plank:

$$\int_0^L \left[p_B(x') - p_T(x')\right] \big(x' - L/2\big) - F_{hinge}\frac{L}{2}\cos\theta = 0. \tag{3}$$

[1]Strictly speaking this is not exactly correct unless there is a second plank, as indicated in light grey in Figure 1, preventing flow above centerline from circulating below the plank.

Following two streamline, one that passes just above the plank and one just below, we can use Bernoulli to find p_B and p_T:

$$p_{in} + \frac{1}{2}\rho v_{in}^2 = p_T(x) + \frac{1}{2}\rho v_T^2(x) = p_B(x) + \frac{1}{2}\rho v_B^2(x) \tag{4}$$

Hence

$$p_B(x) - p_T(x) = \frac{1}{2}\rho\left[v_T^2(x) - v_B^2(x)\right] \tag{5}$$

Using conservation of mass and $y_T(x) = R - x\tan\theta$ and $y_B(x) = R + x\tan\theta$ we find

$$v_T^2(x) - v_B^2(x) = \frac{Q^2}{4}\left(\frac{1}{y_T^2} - \frac{1}{y_B^2}\right) \tag{6}$$

For small θ, $\tan\theta \approx \theta$ and $(1+a\theta)^n \approx 1 + na\theta$ (note this is also true for $n < 0$). So after a little bit of algebra we find:

$$v_T^2(x) - v_B^2(x) \approx \frac{Q^2\theta}{R^3}x \tag{7}$$

Inserting this in equation (2)

$$mg + F_{hinge} = \int_0^L \frac{1}{2}\rho\frac{Q^2\theta}{R^3}x'\,dx' = \frac{Q^2L^2\rho}{4R^3}\theta. \tag{8}$$

Similarly, from equation (3)

$$F_{hinge} = -\frac{1}{12}\frac{\rho Q^2L^2}{R^3}\theta. \tag{9}$$

Hence

$$\boxed{\theta_{eq} \approx \frac{3mgR^3}{\rho Q^2L^2}} \tag{10}$$

(Actually this calculation is even easier if you take the torque balance about the hinge ... then you don't need to apply force balance since you don't need F_{hinge} in the torque balance equation.)

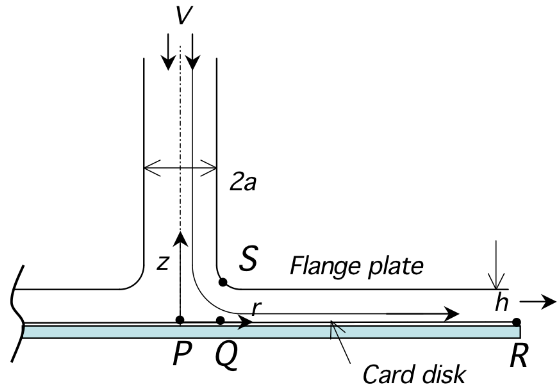
The lift in this problem is supplied because the flow above the plank must speed up as it goes through the narrowing gap. Thus, this equilibrium is $\boxed{\text{UNSTABLE}}$ since, if θ is increased, the flow above the plank must go even faster, further lowering the pressure, amplifying the perturbation. Likewise, if θ is decreased, the flow above slows down, again amplifying the perturbation.

Now we can also address the stability of the vertical equilibrium position. We have chosen the weight of the plank and the flow rate of the fluid s.t. $\theta_{eq} \ll 1$. Thus, if we consider a position $\theta > \theta_{eq}$, we know that the torques arising from the flow will beat the torques due to gravity. Hence, the vertical plank is $\boxed{\text{STABLE}}$.

(1 point.) Comment on other equilibria positions using symmetry arguments (these may vary depending on what assumptions you made earlier).

2. Lift force on a circular disk

Photo removed for copyright reasons.
See film "Pressure Fields and Fluid Acceleration"
at http://web.mit.edu/fluids/www/Shapiro/ncfmf.html



Inflow: $Q = \pi a^2 V = \text{constant}$

Model as 1D (radial) outflow of fluid;
constant density, inviscid steady flow

$$\underline{v} = \begin{Bmatrix} v_r \\ v_\theta \\ v_2 \end{Bmatrix} = \begin{Bmatrix} f(r) \\ 0 \\ 0 \end{Bmatrix}$$

(a) Point P is a stagnation point with maximum pressure: $p(r=0) = p_{stag} = p_0 + \frac{1}{2}\rho V^2 + (\rho g H)$ (henceforth we ignore the small hydrostatic term).

In the outer region $(r > a)$ we have pure radial flow; no centripetal acceleration, no acceleration across streamlines; hence $\dfrac{\partial P}{\partial n} = \dfrac{\partial p}{\partial \theta} = 0$. Pressure field $\neq$ function of θ.
In the inner flow the streamlines are curved; there is a pressure gradient normal to streamlines since pressure decreases towards center of curvature; we therefore expect: $p_s < p_Q < p_P$; Stagnation pressure is maximum value in system. Hence, since p_s is small we may actually get separation here and a recirculating region (analogous to the *vena contracta* in a jet; viscous effects may play a role here).

(b) Velocity Field: $V_r = \begin{cases} C_1 r & r \le a \\ C_2/r & r \ge a \end{cases}$

The second condition arises because for $r > a$ a mass balance gives: $\pi a^2 V = Q = 2\pi h v_r$

or $v_r = a^2 V / 2hr$ hence $\Rightarrow C_2 = \dfrac{a^2 V}{2h}$

Equating velocities at radius $r = a$ gives $C_1 a = \dfrac{a^2 V}{2ha}$ $\Rightarrow C_1 = \dfrac{V}{2h}$

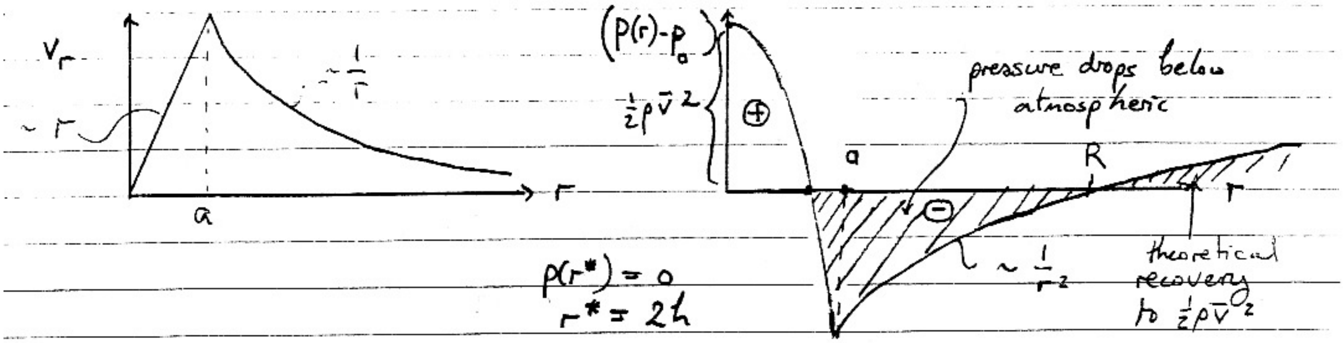
(c) Pressure Field

Applying Bernoulli Equation along streamline from $\overrightarrow{PQR}$ (ignoring $\rho g z$ term)

$$p_T = p_0 + \tfrac{1}{2}\rho V^2 = p(r) + \tfrac{1}{2}\rho\left[v_r(r)\right]^2$$

or by substituting for v_r in each region ($r < a$, $r > a$) we find:

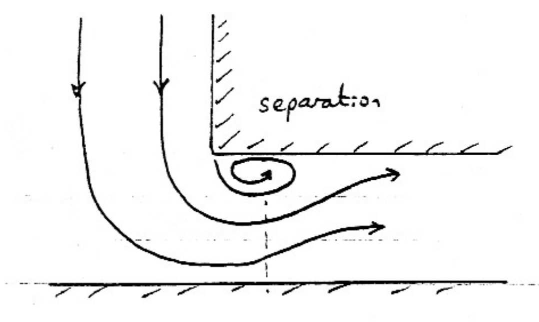
$$p(r) - p_0 = \begin{cases} \dfrac{1}{2}\rho V^2\left[1 - r^2/4h^2\right] & r \le a \\ \dfrac{1}{2}\rho V^2\left[1 - a^4/4h^2r^2\right] & r \ge a \end{cases}$$



In fact on deeper consideration, you can see that this entire analysis is really only valid (or plausible) for $h \le a/2$, otherwise the proposed velocity field and pressure field do not even make sense.

In outer region ($r > a$) we expect $v_r \downarrow$ as $r \uparrow$ and hence that $p(r) \uparrow$ as $r \uparrow$ and approaches $p_\infty = p_0$ at $r \to \infty$. In the inner region, the stagnation pressure $p_{stag} = p_0 + \dfrac{1}{2}\rho V^2 \left(+\rho g z << 1\right)$ is the largest pressure and then velocity decreases and pressure decreases…

However, if $h \ge a/2$ then we cannot reconcile these two variations in $p(r) \Rightarrow$ The analysis is wrong because of separation of the stream near the sharp corner and the resulting formation of vortices. The role of viscosity is to smooth the sharp gradients in velocity close to $r \approx a$ and adjust the pressure field so that for a given radius plate, the pressure field smoothly recovers to p_0 at R.



(d) Total Force on Disk

Clearly the pressure decreases quadratically with r in the inner region and the gage pressure at $r = a$ becomes negative if:

$$1 - a^2/4h^2 < 0 \qquad \text{or} \qquad h \le \tfrac{1}{2}a \qquad (\text{i.e. } a \ge 2h)$$

There is then a region where pressure acting on disk is upwards. This "lift" may overcome the weight of the disk as V increases. Note that the result requires use of *gage* pressure (because p_0 acts on both sides). The pressure force downwards on the disk is given by with element of area $\{dA = 2\pi r dr\}$

$$F_P = \int_0^R p(r)2\pi r dr = \int_0^a \pi\rho V^2\left[r - r^3/4h^2\right]dr + \int_a^R \pi\rho V^2\left[r - \frac{a^4}{4h^2 r}\right]dr$$

$$= \pi\rho V^2 \left[\frac{1}{2}r^2 - \frac{r^4}{16h^2}\right]_0^a + \pi\rho V^2 \left[\frac{1}{2}r^2 - \frac{a^4}{4h^2}\ln r\right]_a^R$$
$$= \pi\rho V^2 \left[\frac{1}{2}R^2 - \frac{a^4}{16h^2}\right] + \pi\rho V^2 \left(\frac{a^4}{4h^2}\right)\ln\left(a/R\right)$$

or finally $F_P = \frac{1}{2}\pi\rho V^2 \left[\underset{(1)}{R^2} - \underset{(2)}{\frac{a^4}{8h^2}}\right] + \underset{(3)}{\pi\rho V^2 \left(\frac{a^4}{4h^2}\right)\ln\left(a/R\right)}$

Note that
- i) since $a/R < 1$, term 3 is negative
- ii) First term 1 is always $+ve$
- iii) Term 1 – 2 changes sign when: $\frac{a^4}{8h^2} \geq R^2$ i.e. $h \leq \sqrt{\frac{a^4}{8R^2}} = \frac{a^2}{2\sqrt{2}R}$

For Given Conditions h = 0.1 cm, a = 1.0 cm, R = 5 cm; the critical radius is therefore: $r^* = 2h = 0.2cm < a$; hence pressure does go negative in inner region.

For lift we require $F_P + W = 0$
$-ve$ lift force from sub atmospheric pressure $+ve$ gravity force (10 grams) = 0
$\rho_{air} = 1.2kg/m^3 = 1.2\times10^{-3} g/cm^3$

$$\frac{1}{2}\pi\left(1.2\times10^{-3}\right)V^2\left[5^2 - \frac{1^4}{8\times0.01}\right] + \pi\left(1.2\times10^{-3}\right)\left(\frac{1^4 V^2}{4\times0.01}\right)\ln\left(1/5\right) = -10$$

$$0.0236V^2 - 0.1517V^2 = -10 \text{ or } V = 8.84 \text{ m/s}$$

(e) Unsteady flow: the Bernoulli equation applied between points P (where the stagnation pressure is given by eq. ( ) and point R gives

$$\int_P^R \rho\frac{\partial v_r(r,t)}{\partial t}ds + \left[p(R) + \tfrac{1}{2}\rho v_R{}^2\right] - \left[p_{stag} + 0 + 0\right] = 0$$

The only thing that is varying in this problem is the gap *h(t)*. We use conservation of mass with a cylindrical control volume of radius r and height *h(t)*. We can then derive an expression for the change in velocity with time as follows;

$$\frac{dM_{sys}}{dt} = 0 = \frac{dM_{cv}}{dt} + \int_A \rho\left(\boldsymbol{v} - \boldsymbol{v}_c\right)\cdot\mathbf{n}dA \text{ which gives (in each separate region):}$$

$$0 = \frac{d}{dt}\left(\pi r^2 h\right) - \pi r^2 V + 2\pi rh\ v_r \text{ for } r \leq a \qquad \text{and} \qquad 0 = \frac{d}{dt}\left(\pi r^2 h\right) - \pi a^2 V + 2\pi rh\ v_r \text{ for } r \geq a$$

$$\text{and hence } v_r(r,t) = \begin{cases} \dfrac{Vr}{2h(t)} - \dfrac{r}{2h(t)}\dfrac{dh}{dt} & r \le a \\ \dfrac{a^2V}{2rh(t)} - \dfrac{r}{2h(t)}\dfrac{dh}{dt} & r \ge a \end{cases}$$

Note that $\dot{h} = dh/dt < 0$ so the velocity is increased, and also note that if $\dot{h} = 0$ then these expressions for the velocities reduce to expressions in part (b).
We substitute these expressions into the integral term, and break it down into two parts (integrated along a streamline *ds = dr)*. The unsteady Bernoulli equation then becomes:

$$\int_0^a \frac{\rho}{2}\left[V\left(-\frac{\dot{h}}{h^2}\right) + \left\{\frac{\dot{h}^2}{h^2} - \frac{\ddot{h}}{h}\right\}\right] rdr + \int_a^R \frac{\rho}{2}\left[\frac{a^2V}{r}\left(-\frac{\dot{h}}{h^2}\right) + \left\{\frac{\dot{h}^2}{h^2} - \frac{\ddot{h}}{h}\right\} r\right] dr + \left[p_0 + \tfrac{1}{2}\rho {v_R}^2\right] - p_R = 0$$

Assuming that $p_R = p_{stag} = p_0 + \frac{1}{2}\rho V^2$ and $v_R = a^2V/2Rh(t) - R\dot{h}/2h(t)$ we obtain:

$$\frac{R^2}{2}\left\{\frac{\dot{h}^2}{h^2} - \frac{\ddot{h}}{h}\right\} - a^2V\left(\frac{1}{2} + \ln\frac{R}{a}\right)\frac{\dot{h}}{h^2} + \frac{1}{4}\left[\frac{a^2V}{Rh} - \frac{R}{h}\dot{h}\right]^2 - V^2 = 0$$

so the gap separation decreases with time in a rather complex nonlinear way! (This equation can be solved as an Initial Value Problem with conditions $h(t=0) = h_0; \dot{h}(t=0) = 0$).