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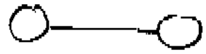
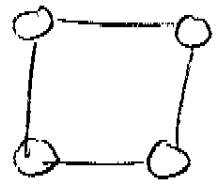
Hypercube network

d dimensions
$N = 2^d$ nodes

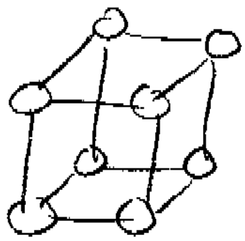
$d=0$
$N=1$



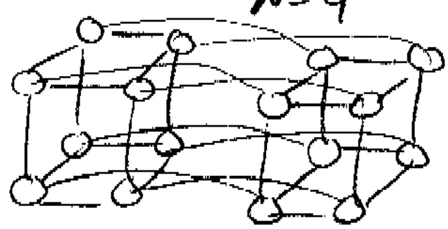
$d=1$
$N=2$



$d=2$
$N=4$



$d=3$
$N=8$



$d=4$
$N=16$

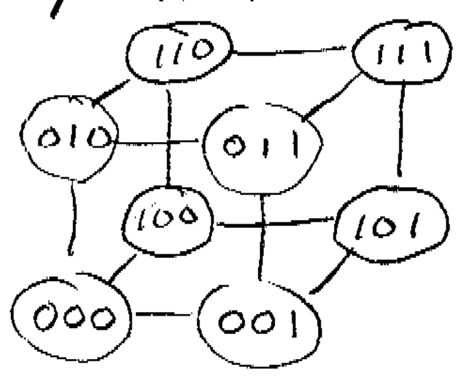
Label each of the 2^d nodes with a d-bit binary string:

$b_{d-1}\, b_{d-2} \cdots b_0$

Connect two nodes if they differ in exactly 1 bit: (Hamming distance = 1)

\# bit pos in which they differ.

$b_{d-1}\, b_{d-2} \ldots b_0$
connected to
$\overline{b_{d-1}}\, b_{d-2} \ldots b_0$
$b_{d-1}\, \overline{b_{d-2}} \ldots b_0$
$\vdots$
$b_{d-1}\, b_{d-2} \ldots \overline{b_0}$.



Diameter $= d = \lg N$
Degree $= d = \lg N$
BW $= N/2$
\#wires $= Nd/2 = \Theta(N \lg N)$

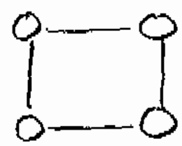
Embeddings in the hypercube

Theorem The N-node hypercube contains an N-node linear array as a subgraph (i.e., a hamiltonian path).

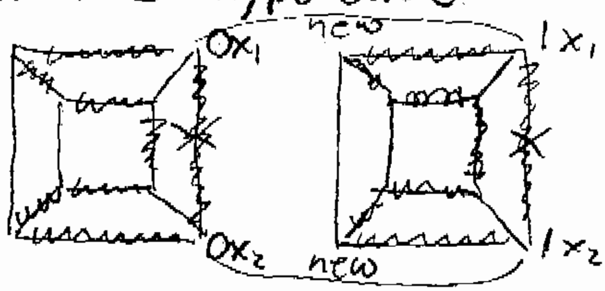
Pf. True for $N = 1, 2, 4$:

Induction on d. Claim $\exists$ hamiltonian cycle for d-dim hypercube for $d \geq 2$.

Base:



Assume claim true for $N/2$-node hypercube.
Consider $N = 2^d$ hypercube



Consists of 2 $N/2$-node hypercubes containing (identical) hamiltonian cycles (by IH). Let $(0x_1, 0x_2)$ be any edge in 1st subcube that cycle goes through, and let $(1x_1, 1x_2)$ be corresp. edge in 2nd subcube. Replace these two edges with $(0x_1, 1x_1)$ and $(0x_2, 1x_2)$. $\boxtimes$.

<u>Def.</u> A d-bit <u>Gray code</u> is an ordering of the 2^d d-bit bit-strings such that each string differs from the previous in exactly one bit.

<u>Ex.</u> $d=3$

0	000
1	001
2	011
3	010
4	110
5	111
6	101
7	100

"Reflecting" Gray code

<u>Corollary.</u> d-bit Gray codes exist $\forall d$. $\boxtimes$

Hamiltonian path in hypercube = Gray code.

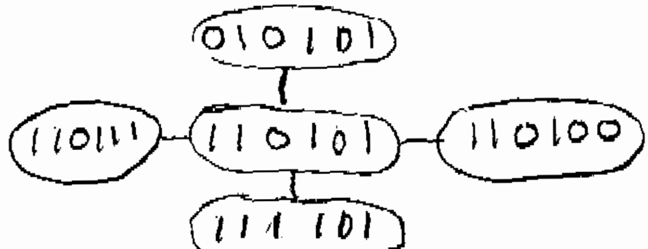
<u>Theorem.</u> Let $d_1 + d_2 = d$. Then a $2^{d_1} \times 2^{d_2}$ mesh (or torus) can be embedded in an $N = 2^d$-node hypercube.

Pf. Let $g_1(x_1)$ be d_1-bit Gray code of x_1, where $0 \le x_1 < 2^{d_1}$.
$g_2(x_2)$ d_2 x_2 $0 \le x_2 \le 2^{d_2}$.

Map node (x_1, x_2) of mesh to node $g_1(x_1) \| g_2(x_2)$ of hypercube. $\boxtimes$
↑ concatenate

Ex. 8 × 8 mesh

(3,6)
|
(4,5) — (4,6) — (4,7)
|
(5,6)



<u>Corollary.</u> $2^{d_1} \times 2^{d_2} \times \cdots \times 2^{d_k}$ mesh embedded in $2^{d_1 + d_2 + \cdots + d_k}$ hypercube.

Fact: 3×5 mesh <u>cannot</u> be embedded in 16-node hypercube.
But, $m \times n$ mesh <u>can</u> be embedded in $2^{\lceil \lg(mn) \rceil}$-node hypercube with dilation 2.

Embedding trees in hypercubes

Thm. Not possible to embed $(N-1)$-node complete binary tree in N-node hypercube.

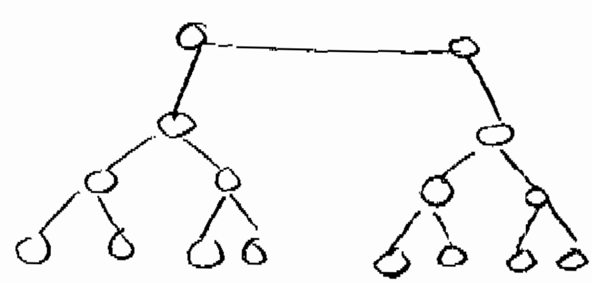
Proof. Sup. possible. Root mapped to node $00\ldots0$.
Depth-1 nodes mapped to nodes with odd parity.
Depth-2 " " " " " even "
Depth-3 " " " " " odd "

(**Def.** Parity $= \begin{cases} \text{odd if \# 1's is odd} \\ \text{even if \# 1's is even} \end{cases}$)

\# leaves $= N/2$: all have same parity
\# grandparents of leaves $= N/8$: same parity as leaves.

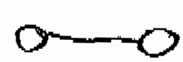
But, hypercube has $N/2$ nodes with even parity and $N/2$ nodes with odd parity, and tree must have $\geq N/2 + N/8$ nodes with same parity. #

Def. Double-rooted complete binary tree:

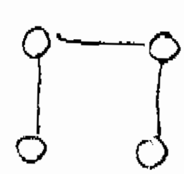


Thm. N-node double-rooted cbt is subgraph of N-node hypercube, for $N \geq 2$.

Proof. Induction on d.
$d = 1$ $(N = 2)$:



$d = 2$ $(N = 4)$



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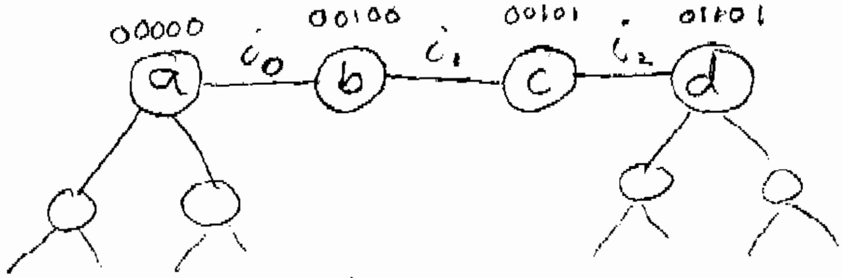
$d \geq 3$ $(N \geq 8)$: Embed double-rooted cbt on $N/2$ nodes in $N/2$-node 0-subcube. Consider top 4 nodes:

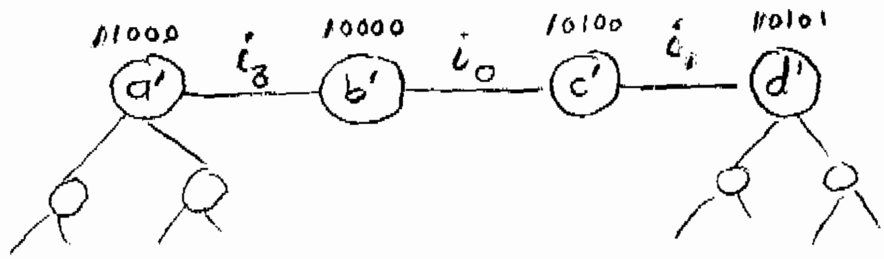


WLOG, $a = 00\cdots0$ ← 0-subcube

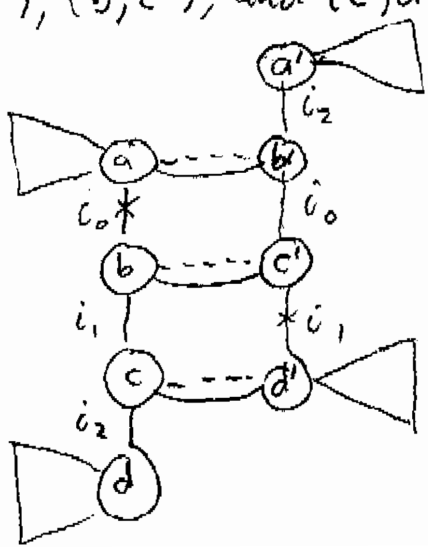
a, b differ in dim i_0
b, c " " " i_1
c, d " " " i_2

Note: $i_0 \neq i_1 \neq i_2$ (or else $a = c$ or $b = d$).

Embed double-rooted cbt on $N/2$ nodes in $N/2$-node 1-subcube identically, except $b' = 10 0\ldots0$ and permute dimensions $\$$ $i_1 \to i_0$ and $i_2 \to i_1$.



Thus, (a, b'), (b, c'), and (c, d') adjacent.



b, c' new roots

$\boxtimes$

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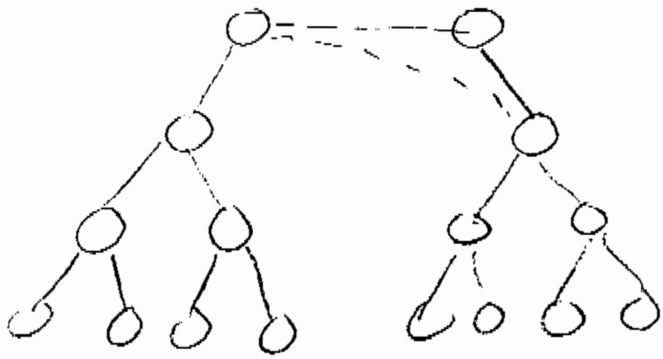
Corollary (N-1)-node CBT embeds in N-node hypercube with dilation 2.

Pf.



Embed CBT into double-rooted CBT with 1 edge having dilation 2. $\boxtimes$

Fact: All N-node binary trees can be embedded into N-node hypercube with O(1) dilation.
$\hookrightarrow \leq 5$