Solution set 1

Warmups

Warmup problems are quick problems for you to check your understanding; ***don't turn them in.***

1. Use easy cases to find (in terms of n) the final term in the sum

$$S = \underbrace{1 + 4 + 9 + \cdots}_{n \text{ terms}}.$$

The pattern is that each term is a perfect square k^2. However, is the final term n^2, $(n+1)^2$, or $(n-1)^2$? Try the extreme case $n = 1$. With only one term, the final (and also first) term is 1, which is n^2. So n^2 is the final term.

2. Find

$$\int_0^\infty e^{-ax}\,dx$$

using dimensions, and then check your answer using easy cases.

Let x be a length (choosing it to be a time would work too). Then the integral is also a length. The exponent ax has to be dimensionless, so $[a] = \mathrm{L}^{-1}$. The integral is therefore a dimensionless constant times $1/a$, because $1/a$ is the only way to construct a length. The dimensionless constant is 1 to match the easy case

$$\int_0^\infty e^{-x}\,dx = 1.$$

The result $1/a$ also passes the easy cases of $a \to \infty$ and $a \to 0$. When $a \to 0$, for example, the integrand is 1, integrated from 0 to ∞, so the integral should be ∞. Indeed, $1/a$ correctly produces ∞.

3. In terms of the dimensions L, M, and T, find the dimensions of energy, power, and density.

Energy is force times distance, and force is $\mathrm{MLT^{-2}}$. So energy is $\mathrm{ML^2T^{-2}}$.

Power is energy per time so it is $\mathrm{ML^2T^{-3}}$.

Density is mass per volume, so it is $\mathrm{ML^{-3}}$.

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4. What are the dimensions of $\partial \mathbf{v}/\partial t$, where $\mathbf{v}$ is the velocity vector?

A vector has the same dimensions as its magnitude. A partial derivative, as far as dimensions are concerned, has the same effect as an ordinary derivative. So

$$\left[\frac{\partial \mathbf{v}}{\partial t}\right] = \frac{[v]}{\mathrm{T}} = \mathrm{LT}^{-2},$$

which are the dimensions of acceleration.

5. What are the dimensions of pressure p? What are the dimensions of ∇p?

Pressure is force per area, so $[p] = \mathrm{ML}^{-1}\mathrm{T}^{-2}$. The gradient ∇ is a vector partial derivative, where the derivative is a space derivative (rather than a time derivative), so it contributes an inverse length, as would an ordinary spatial derivative. Therefore

$$[\nabla p] = \frac{[p]}{\mathrm{L}} = \mathrm{ML}^{-2}\mathrm{T}^{-2}.$$

6. How many dimensions in total are contained in v (velocity), g (gravitational acceleration), and h (height)? How many dimensionless groups can be formed from these three quantities?

There are two dimensions: L and T. So you can form $3 - 2 = 1$ dimensionless groups, for example v^2/gh.

Problems

Turn in these problems.

7. Prove or disprove $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ as simply as you can.

Disproof. Take $x = y = 1$. The proposed equation gives

$$\sqrt{2} = \sqrt{1} + \sqrt{1},$$

which is junk.

8. Use dimensions to find

$$\int \frac{dx}{x^2 + a^2}.$$

Check your answer in two ways: using easy cases and using a variable substitution.

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A useful intermediate result:

$$\int \frac{dx}{x^2+1} = \arctan x + C.$$

Since the denominator is $x^2 + a^2$, the dimensions of a are the same as the dimensions of x. Pretend that x is a length. Then the integral has dimensions of length (from the dx) divided by length squared (from the $x^2 + a^2$). The integral with $a = 1$ is $\arctan x$, which is dimensionless. To give it dimensions of L^{-1}, multiply by a^{-1}. Thus $a^{-1} \arctan x$ is a reasonable candidate. Except that the argument of an arctangent is, like the argument of an exponential, dimensionless. If x has dimensions of length, then $\arctan x$ is not legal. To fix the dimensions, use x/a instead:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C.$$

As a check on this result, try a few extreme cases. The limit $a \to 0$ is tricky because (1) the integrand becomes $1/x^2$, which blows up when $x \to 0$, and (2) the candidate result contains a^{-1}, which blows up when $a \to 0$, and x/a, which also blows up when $a \to 0$. So to use the $a \to 0$ case you'd have to check that all the infinities are the same kind of infinity. It is easier to try $a \to \infty$. The integrand goes to zero because of the a^2 in the denominator, and the result goes to zero because of the a^{-1}. So all is well. The other extreme (or special) case is $a = 1$. It works fine: With $a = 1$, the candidate result becomes

$$\int \frac{dx}{x^2+1} = \arctan x + C,$$

which I gave as the useful result (so it has a reasonable chance of being true!).

9. Use the Navier–Stokes equations to show that the dimensions of viscosity ν are $\mathrm{L}^2\mathrm{T}^{-1}$. This result was needed when we used dimensions to find the terminal speed of the falling cones.

Every term in the Navier–Stokes equations has identical dimensions, so match the dimensions of the viscosity-containing term $\nu\nabla^2\mathbf{v}$ with the dimensions of another term. The simplest term is the first one, $\partial\mathbf{v}/\partial t$, which is (one part of) the acceleration. The partial derivative ∂ acts just like a d in the sense of meaning 'a little bit of'. And the vector $\mathbf{v}$ has the same dimensions as any of its components, which have dimensions of velocity: LT^{-1}. So

$$\left[\frac{\partial \mathbf{v}}{\partial t}\right] = \mathrm{LT}^{-2}.$$

To find the dimensions of $\nabla^2\mathbf{v}$, remember that ∇ is just a fancy (vectorial) spatial derivative. It is therefore like d/dx in how it alters dimensions. So ∇^2 is like $(d/dx)^2$, which divides dimensions by length squared. Therefore the dimensions of $\nabla^2\mathbf{v}$ are

$$\left[\nabla^2\mathbf{v}\right] = \frac{[v]}{\mathrm{L}^2} = \mathrm{L}^{-1}\mathrm{T}^{-1}.$$

When this term is multiplied by the viscosity ν, the dimensions have to match those of acceleration, which are LT^{-2}. So

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$$[\nu] = \frac{\mathrm{LT}^{-2}}{\mathrm{L}^{-1}\mathrm{T}^{-1}} = \mathrm{L}^2\mathrm{T}^{-1}.$$

Those dimensions, by the way, are the dimensions of a **diffusion constant**, indicating that viscosity results from a random walk.

10. Use easy cases to judge these formulas for $\cos(A + B)$:

a. $\cos A \cos B + \sin A \sin B$
b. $\cos A \cos B - \sin A \sin B$
c. $\sin A \cos B + \cos A \sin B$
d. $\cos A \sin B - \sin A \sin B$

To guess the valid candidate for $\cos(A+B)$, try the extreme case $B = 0$. Then $\cos B = 1$ and $\sin B = 0$, so

$$\begin{aligned}
\cos A \cos B + \sin A \sin B &= \cos A\\
\cos A \cos B - \sin A \sin B &= \cos A\\
\sin A \cos B + \cos A \sin B &= \sin A\\
\cos A \sin B - \sin A \sin B &= 0.
\end{aligned}$$

A valid candidate produces $\cos A$, so the third and fourth candidates are wrong. Now try $B = \pi/2$, making $\cos B = 0$ and $\sin B = 1$. Then, of the surviving candidates:

$$\begin{aligned}
\cos A \cos B + \sin A \sin B &= \sin A\\
\cos A \cos B - \sin A \sin B &= -\sin A
\end{aligned}$$

A valid candidate should give $\cos(A + \pi/2) = -\sin A$, so only the second candidate survives. As long as at least one of the candidates was correct (!),

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

11. In this problem you derive a version of Kepler's third law, which connects a planet's orbital period with its distance from the sun.

Planets orbit around the sun mostly according to Newton's law of gravitation. This law says that the acceleration of the planet due to the sun's gravity is k/r^2, where r is the distance from the sun and k is a constant. To keep the analysis simple, assume that the orbit is a circle so that r is constant. So the quantities of interest are the period T, the orbital radius r, and the constant k.

a. What are the dimensions of k? How many dimensions do the three quantities T, r, and k contain in total?

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Acceleration has dimensions LT^{-2}. To make k/r^2 be an acceleration, $[k]$ has to be $\mathrm{L}^3\mathrm{T}^{-2}$.

The three quantities contain 2 dimensions (L and T).

b. Show that you can form only one dimensionless group from T, r, and k.

By the Buckingham Pi theorem, you can form $3 - 2 = 1$ (independent) dimensionless groups.

c. There are many choices for that group. Make any reasonable choice and thereby find a proportionality relation between T and r.

One possibility is kT^2/r^3. Since there are no other (independent) groups, the most general dimensionless statement is $kT^2/r^3 = \text{constant}$. So

$$T \propto r^{3/2}.$$

12. Imagine racing a single small cone – for example, like the one used in lecture on Friday – against a stack of four such cones. Give a rough value for the ratio

$$\frac{\text{terminal speed of the stack of four cones}}{\text{terminal speed of the single cone}}.$$

At terminal velocity, the drag force matches the weight. So the drag force of the quadruple stack is four times the drag force on the single stack (when both are at their respective terminal velocities).

Using dimensions and easy-cases reasoning, we derived that the drag force is $F \sim \rho v^2 A$. Since both stacks have the same cross-section A and move through the same fluid, with the same density ρ, the only way to change F is to change v. To increase F by a factor of 4, increase v by a factor of 2. The velocity ratio is therefore 2 (as we found by experiment in lecture).

Bonus problems

*Bonus problems are more difficult but **optional** problems for those who are curious.*

13. For the falling cones, the Reynolds number was much larger than 1. In this problem, you analyze the opposite limit $\mathrm{Re} \ll 1$.

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a. One way to make Re $\ll 1$ is to use a fluid where $\nu \to \infty$. So the low-Reynolds-number limit is also the high-viscosity limit. In that limit, drag is due almost entirely to viscous forces, so the drag force is proportional to viscosity. Using that information, find the form of the function f in

$$\frac{F}{\rho v^2 r^2} \sim f\left(\frac{rv}{\nu}\right),$$

where $\sim$ means that you need not worry about dimensionless constants.

The only way to get $F \propto \nu$ is for $f(rv/\nu)$ to have the form $f(\text{Re}) \sim 1/\text{Re}$, where $\text{Re} = rv/\nu$ is the so-called Reynolds number. Then

$$\frac{F}{\rho v^2 r^2} \sim f\left(\frac{rv}{\nu}\right) \sim \frac{\nu}{rv}.$$

The drag force itself is then linear in the velocity:

$$F \sim \rho \nu v r.$$

This formula is the famous Stokes drag. [Stokes also calculated the missing constant of proportionality when the moving object is a sphere. In that case, the missing constant is 6π.]

b. Then *sketch* (rather than accurately plot) $f(\text{Re})$ on a log–log plot: First sketch its behavior in the extreme cases Re $\ll 1$ and Re $\gg 1$ then draw a smooth interpolation.

The two extreme cases of f are:

$$f(\text{Re}) \sim \begin{cases} \text{Re}^{-1} & \text{Re} \to 0, \\ 1 & \text{Re} \to \infty. \end{cases}$$

The Re $\to \infty$ limit was discussed in lecture and in Section 11.2 of the notes. It is based on the idea that, at high Reynolds number, drag becomes independent of viscosity, so $f(\text{Re})$ must be a constant. The claim of irrelevance is a subtle lie but it gives basically the correct conclusion. On a log–log plot (a Bode plot for the electrical engineers among you), the function $f(\text{Re}) \sim \text{Re}^{-1}$ is a straight line with -1 slope. So f has a minus-one slope on the left side and is flat on the right side (where $f(\text{Re}) \sim 1$). In between it interpolates smoothly between these limits.

c. Estimate the terminal speed of a fog droplet, which has density $\rho \sim 10^3\,\text{kg}\,\text{m}^{-3}$ (fog droplets are water) and size $r \sim 10^{-5}\,\text{m}$.

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For a fog droplet, which is small and probably falls slowly, assume that at its terminal speed it still has $\text{Re} \ll 1$. We can check that assumption at the end of the calculation, but you need to make some assumption in order to start the calculation. Its weight is

$$W \sim \rho_{\text{water}} g r^3$$

and the drag in the $\text{Re} \ll 1$ limit is

$$F \sim \rho_{\text{air}} \nu v r.$$

At its terminal speed, the drag and weight balance:

$$\rho_{\text{water}} g r^3 \sim \rho_{\text{air}} \nu v r$$

So

$$v \sim \frac{\rho_{\text{water}}}{\rho_{\text{air}}} \frac{g r^2}{\nu}.$$

Let's check the dimensions to catch any mistakes early. The dimensions of gr^2 are L^3T^{-2}, so

$$\left[\frac{g r^2}{\nu}\right] = \frac{\text{L}^3\text{T}^{-2}}{\text{L}^2\text{T}^{-1}} = \text{LT}^{-1},$$

which is a velocity. Good. To get a value for v, put in values for the densities and for g, r, and ν:

$$v \sim \frac{\overbrace{10^3\,\text{kg}\,\text{m}^{-3}}^{\rho_{\text{water}}}}{\underbrace{1\,\text{kg}\,\text{m}^{-3}}_{\rho_{\text{air}}}} \times \underbrace{10\,\text{m}\,\text{s}^{-2}}_{g} \times \frac{\overbrace{10^{-10}\,\text{m}^2}^{r^2}}{\underbrace{10^{-5}\,\text{m}^2\,\text{s}^{-1}}_{\nu}} \sim 0.1\,\text{m}\,\text{s}^{-1}.$$

In a low-lying fog, say 1000 m high, a fog droplet would need 10^4 s, or a few hours, to settle to the ground. Which is how a fog lingers for hours before collecting at the bottom of valleys.

Let's not forget to check that $\text{Re} \ll 1$, which was the initial assumption that we made in order to decide the drag force:

$$\text{Re} \sim \frac{10^{-5}\,\text{m} \times 0.1\,\text{m}\,\text{s}^{-1}}{10^{-5}\,\text{m}^2\,\text{s}^{-1}} \sim 0.1.$$

So $\text{Re} \ll 1$ is not a great approximation, but it is reasonable and no worse than neglecting the factors of $4\pi/3$ in the volume, for example.

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