Problem set 1

Turn in Monday 2008-01-14 at the beginning of class. You need to turn in only the 'Problems' section. The others are for your own practice.

Warmups

Warmup problems are quick problems for you to check your understanding; ***don't turn them in.***

1. Use easy cases to find (in terms of n) the final term in the sum

$$S = \underbrace{1 + 4 + 9 + \cdots}_{n \text{ terms}}.$$

2. Find

$$\int_0^\infty e^{-ax}\, dx$$

using dimensions, and then check your answer using easy cases.

3. In terms of the dimensions L, M, and T, find the dimensions of energy, power, and density.

4. What are the dimensions of $\partial \mathbf{v}/\partial \mathrm{t}$, where $\mathbf{v}$ is the velocity vector?

5. What are the dimensions of pressure p? What are the dimensions of ∇p?

6. How many dimensions in total are contained in v (velocity), g (gravitational acceleration), and h (height)? How many dimensionless groups can be formed from these three quantities?

Problems

Turn in these problems.

7. Prove or disprove $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ as simply as you can.

8. Use dimensions to find

$$\int \frac{dx}{x^2 + a^2}.$$

Check your answer in two ways: using easy cases and using a variable substitution.

A useful intermediate result:

$$\int \frac{dx}{x^2 + 1} = \arctan x + C.$$

9. Use the Navier–Stokes equations to show that the dimensions of viscosity ν are $\mathrm{L}^2\mathrm{T}^{-1}$. This result was needed when we used dimensions to find the terminal speed of the falling cones.

10. Use easy cases to judge these formulas for $\cos(A+B)$:

a. $\cos A \cos B + \sin A \sin B$

b. $\cos A \cos B - \sin A \sin B$

c. $\sin A \cos B + \cos A \sin B$

d. $\cos A \sin B - \sin A \sin B$

11. In this problem you derive a version of Kepler's third law, which connects a planet's orbital period with its distance from the sun.

Planets orbit around the sun mostly according to Newton's law of gravitation. This law says that the acceleration of the planet due to the sun's gravity is k/r^2, where r is the distance from the sun and k is a constant. To keep the analysis simple, assume that the orbit is a circle so that r is constant. So the quantities of interest are the period T, the orbital radius r, and the constant k.

a. What are the dimensions of k? How many dimensions do the three quantities T, r, and k contain in total?

b. Show that you can form only one dimensionless group from T, r, and k.

c. There are many choices for that group. Make any reasonable choice and thereby find a proportionality relation between T and r.

12. Imagine racing a single small cone – for example, like the one used in lecture on Friday – against a stack of four such cones. Give a rough value for the ratio

$$\frac{\text{terminal speed of the stack of four cones}}{\text{terminal speed of the single cone}}.$$

Bonus problems

*Bonus problems are more difficult but **optional** problems for those who are curious.*

13. For the falling cones, the Reynolds number was much larger than 1. In this problem, you analyze the opposite limit $Re \ll 1$.

a. One way to make $Re \ll 1$ is to use a fluid where $\nu \to \infty$. So the low-Reynolds-number limit is also the high-viscosity limit. In that limit, drag is due almost entirely to viscous forces, so the drag force is proportional to viscosity. Using that information, find the form of the function f in

$$\frac{F}{\rho v^2 r^2} \sim f\left(\frac{rv}{\nu}\right),$$

where $\sim$ means that you need not worry about dimensionless constants.

b. Then *sketch* (rather than accurately plot) $f(Re)$ on a log–log plot: First sketch its behavior in the extreme cases $Re \ll 1$ and $Re \gg 1$ then draw a smooth interpolation.

c. Estimate the terminal speed of a fog droplet, which has density $\rho \sim 10^3\,\text{kg}\,\text{m}^{-3}$ (fog droplets are water) and size $r \sim 10^{-5}\,\text{m}$.