6.003: Signals and Systems

Relations between CT and DT:

Insights from Operators and Transforms

February 25, 2010

Mid-term Examination #1

Wednesday, March 3, 7:30-9:30pm.

No recitations on the day of the exam.

Coverage: Representations of CT and DT Systems
Lectures 1–7
Recitations 1–8
Homeworks 1–4

Homework 4 will not collected or graded. Solutions will be posted.

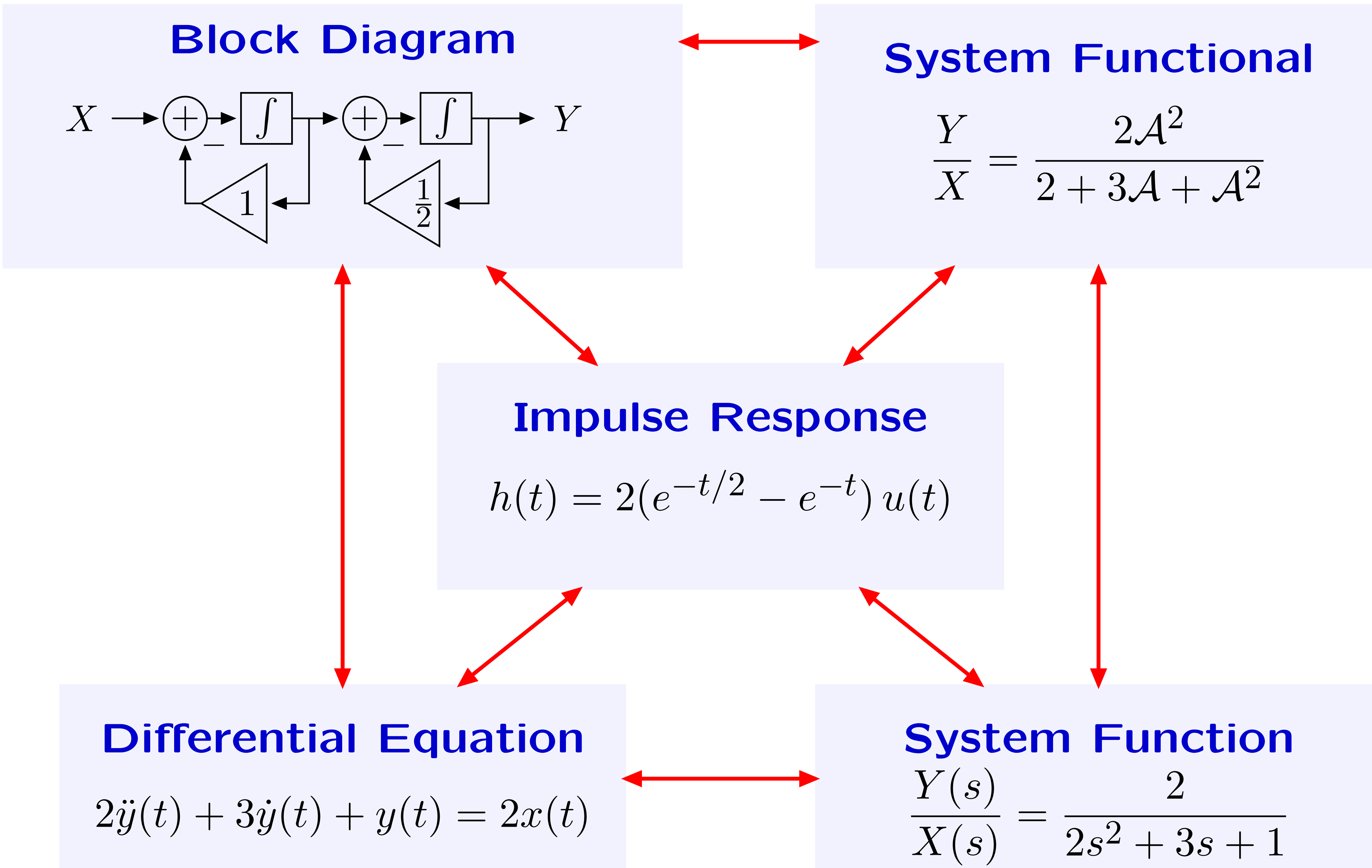
Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

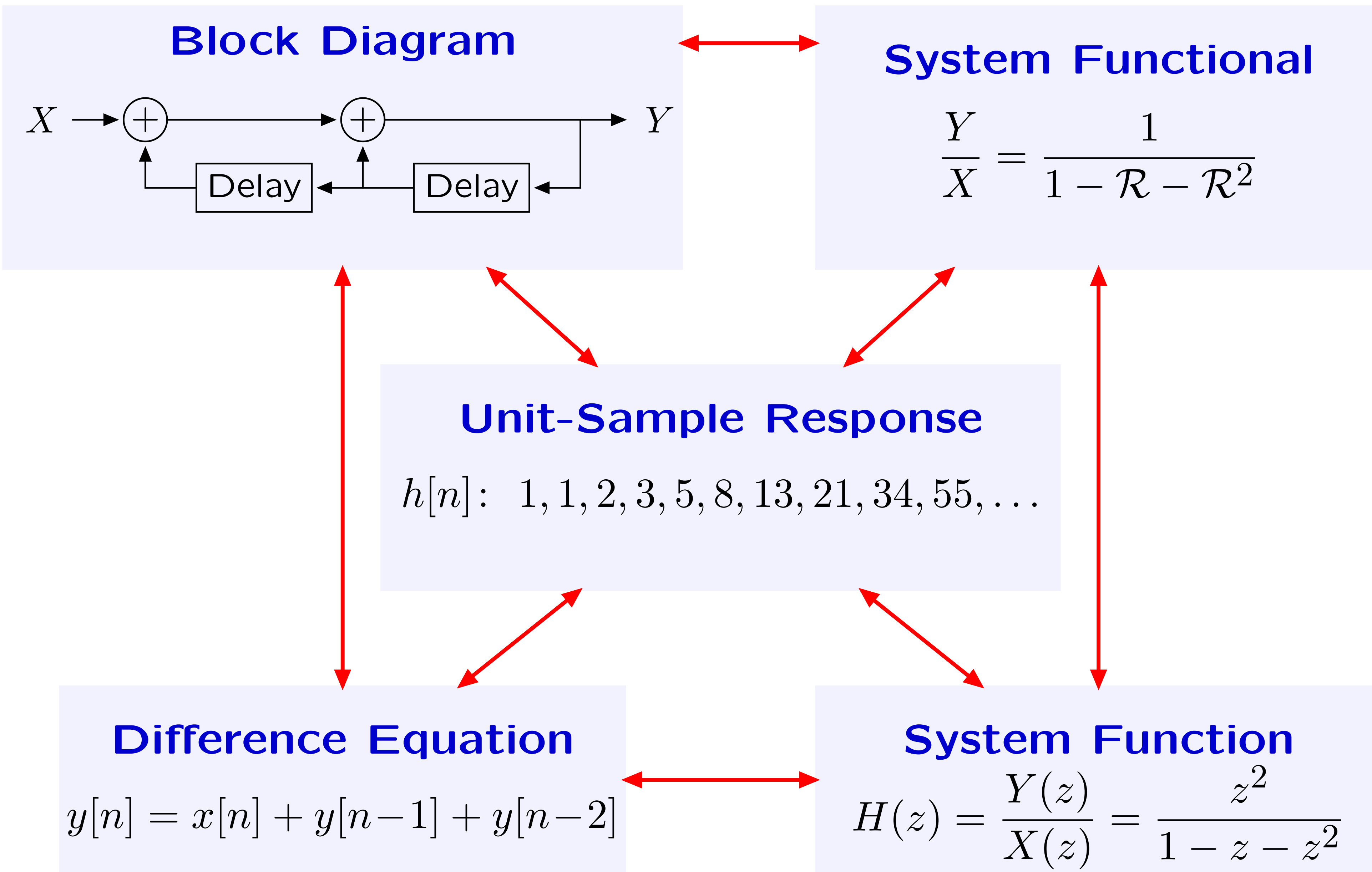
Concept Map: Continuous-Time Systems

Relations among CT representations.



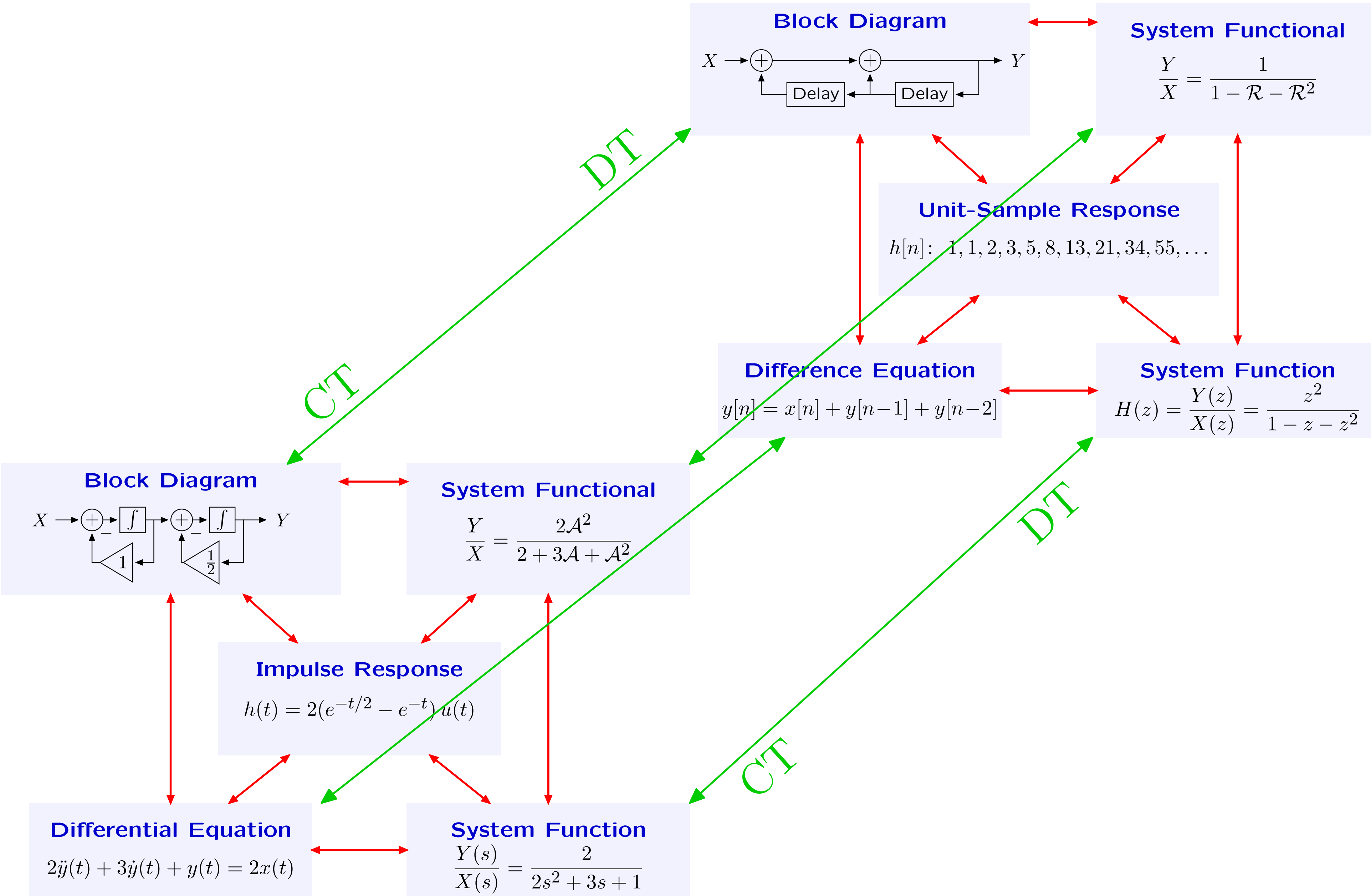
Concept Map: Discrete-Time Systems

Relations among DT representations.



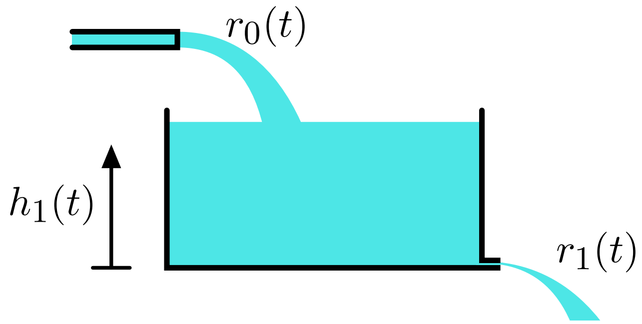
Concept Map

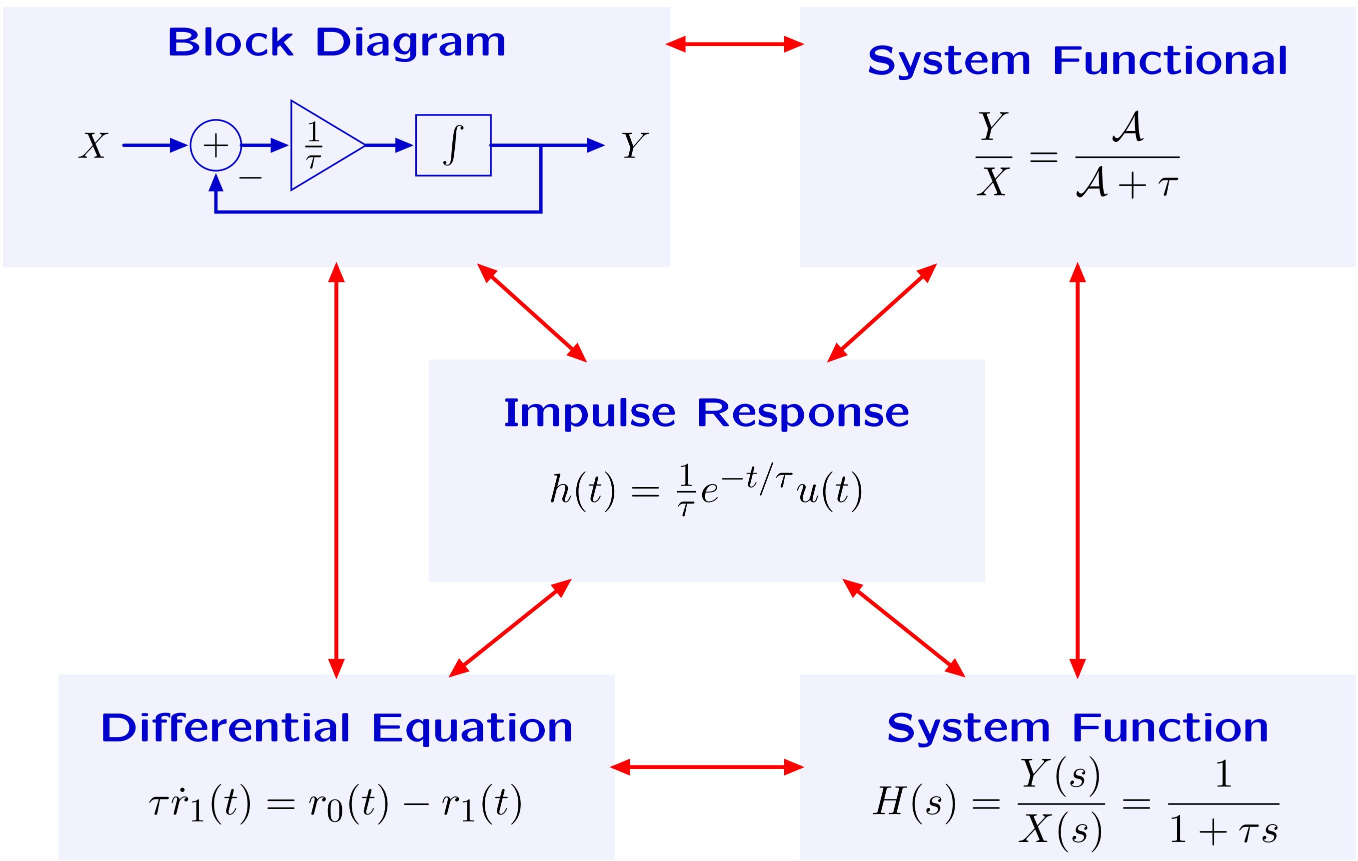
Relations between CT and DT representations.



First-Order CT System

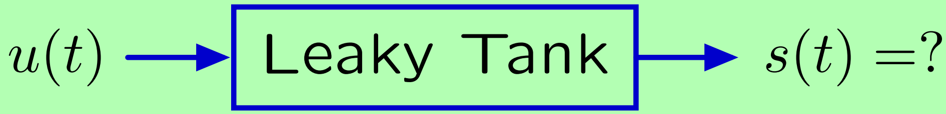
Example: leaky tank.



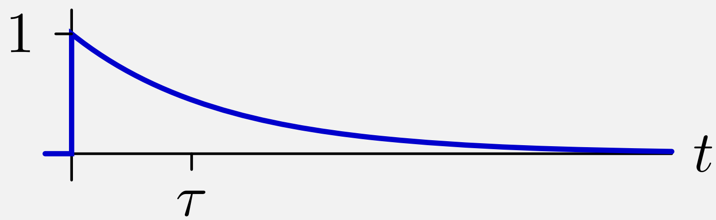


Check Yourself

What is the "step response" of the leaky tank system?



1.



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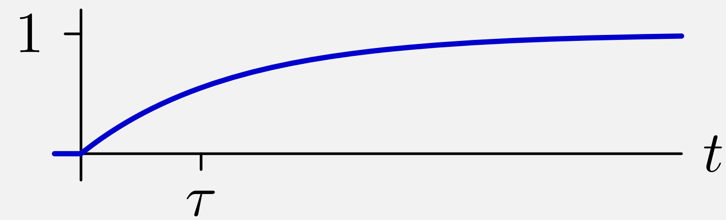

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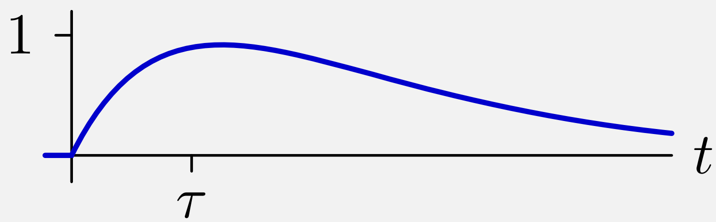

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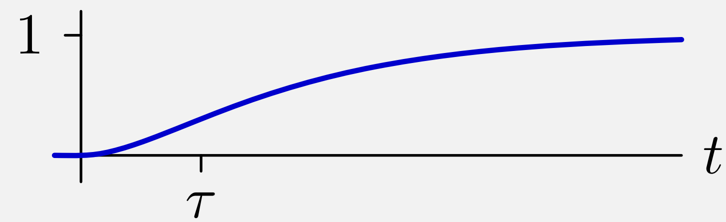
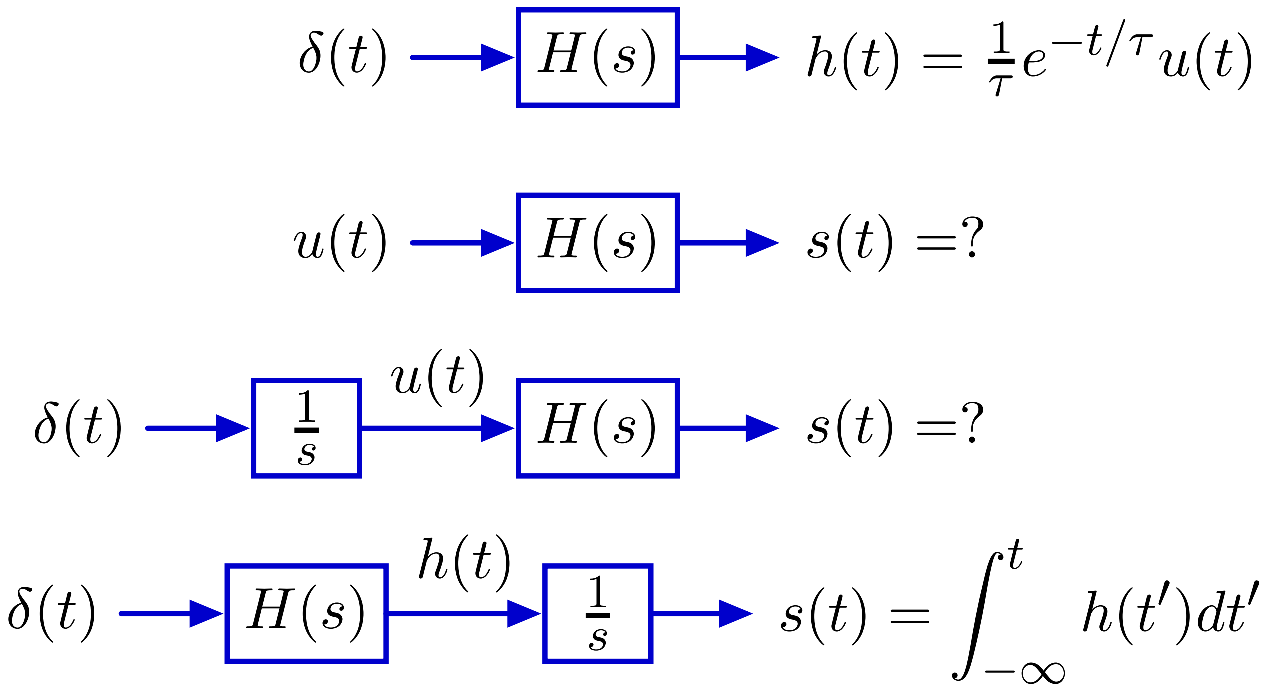


5. none of the above

Check Yourself

What is the "step response" of the leaky tank system?

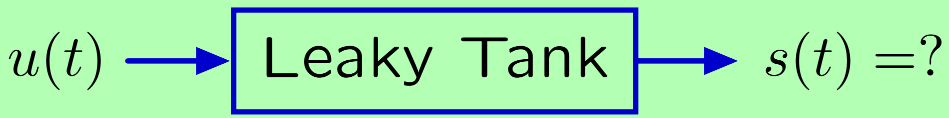


$$s(t) = \int_{-\infty}^{t} \frac{1}{\tau} e^{-t'/\tau} u(t')dt' = \int_{0}^{t} \frac{1}{\tau} e^{-t'/\tau} dt' = \left(1 - e^{-t/\tau}\right) u(t)$$

Reasoning with systems.

Check Yourself

What is the "step response" of the leaky tank system? 2



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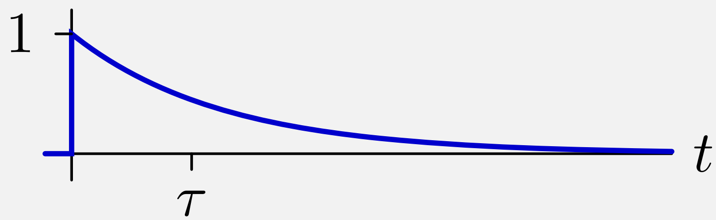

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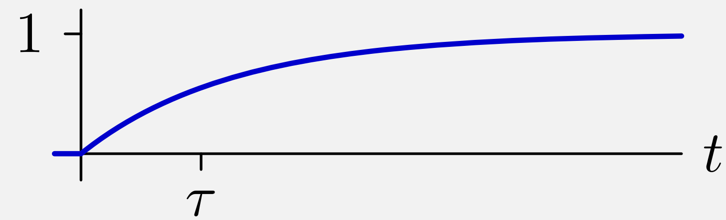

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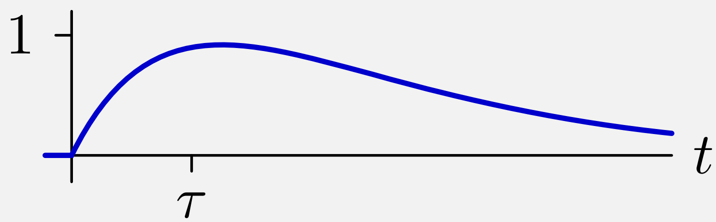

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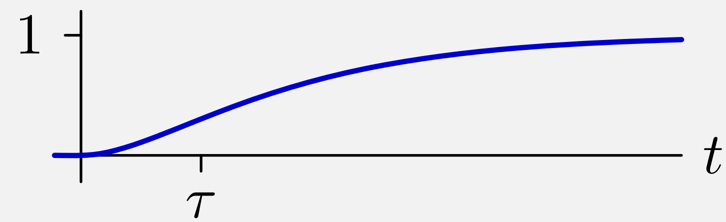


5. none of the above

Forward Euler Approximation

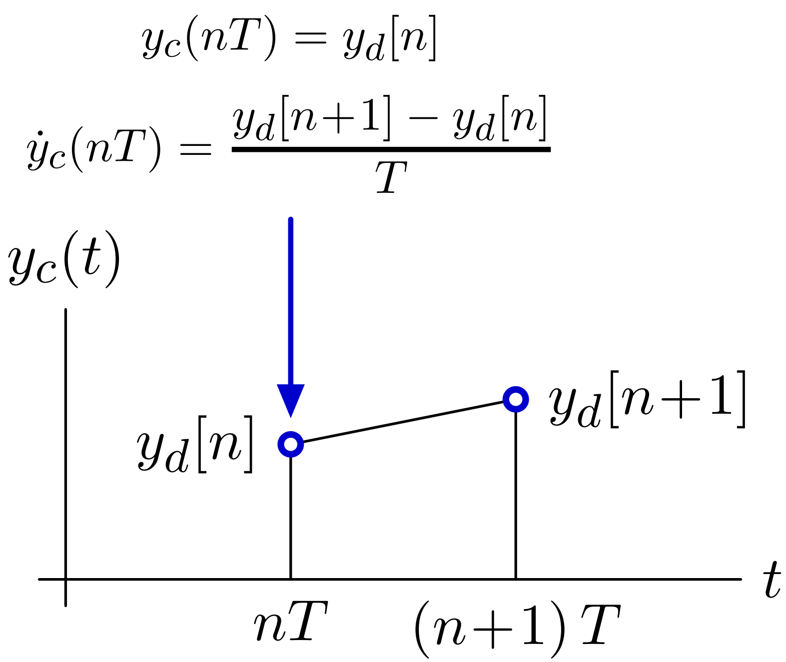
Approximate leaky-tank response using forward Euler approach.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$



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into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

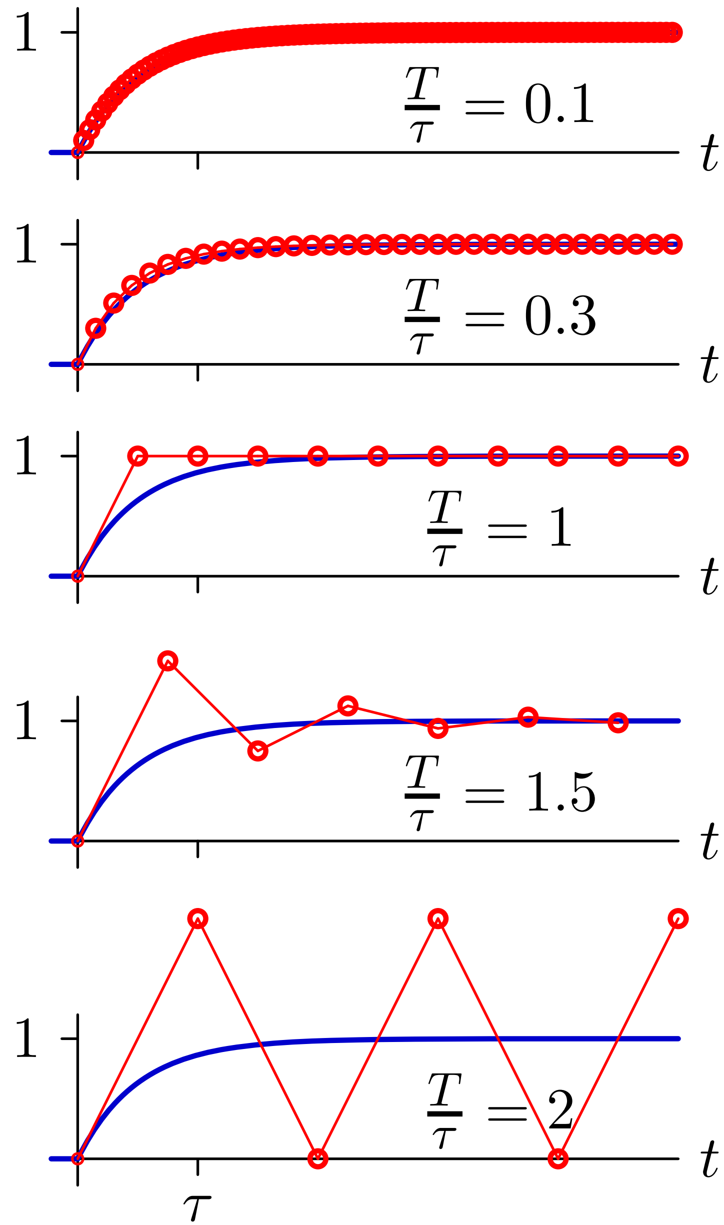
$$\frac{\tau}{T}\Big(y_d[n+1] - y_d[n]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Forward Euler Approximation

Plot.



Why is this approximation badly behaved?

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole.

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Take the Z transform:

$$zY_d(z) - \left(1 - \frac{T}{\tau}\right) Y_d(z) = \frac{T}{\tau} X_d(z)$$

Solve for the system function:

$$H(z) = \frac{Y_d(z)}{X_d(z)} = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

Pole at $z = 1 - \dfrac{T}{\tau}$.

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 2

1. $z = \frac{T}{\tau}$
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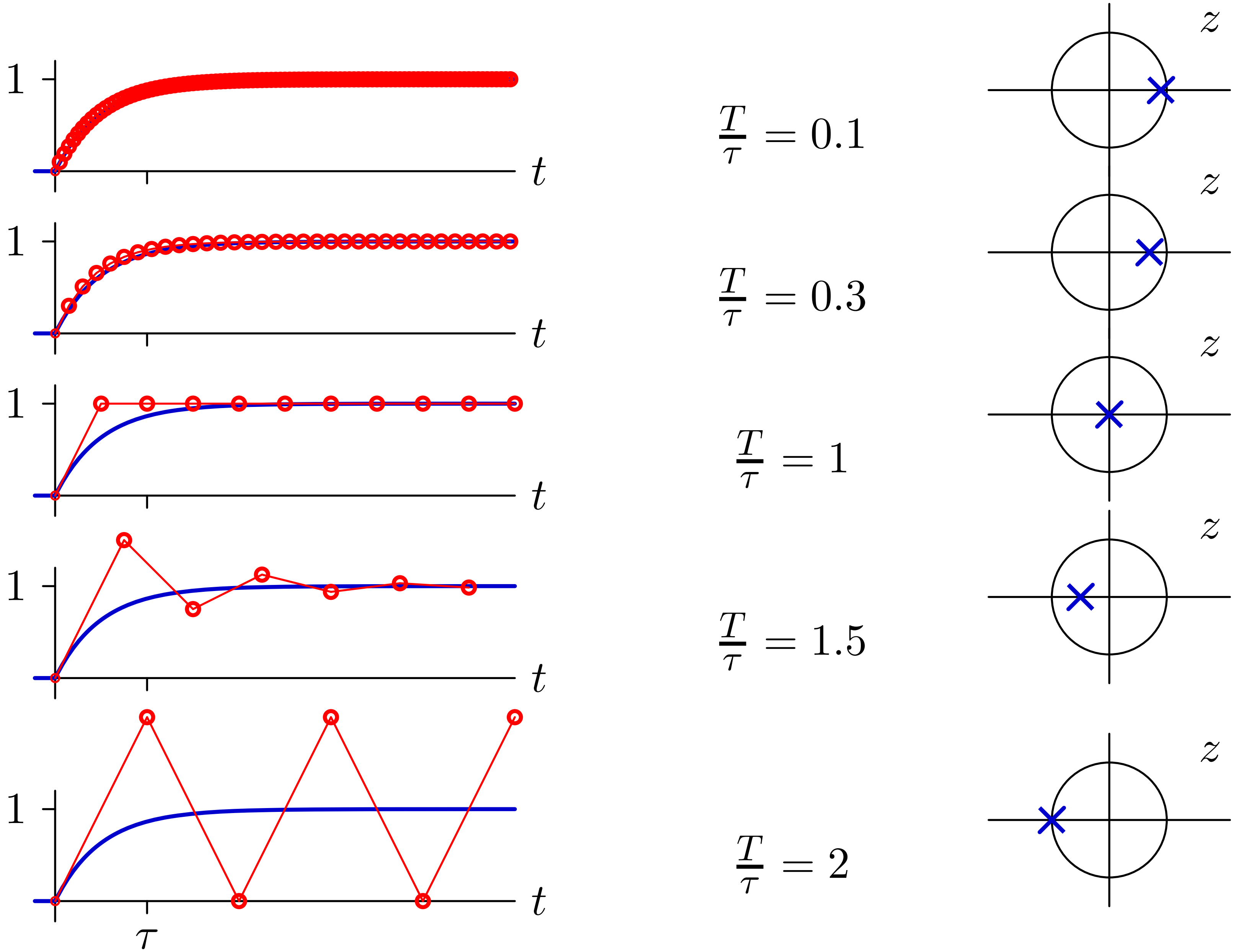
Dependence of DT pole on Stepsize



The CT pole was fixed ($s = -\frac{1}{\tau}$). Why is the DT pole changing?

Dependence of DT pole on Stepsize

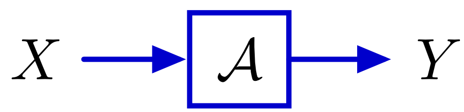
Change in DT pole: problem specific or property of forward Euler?

Dependence of DT pole on Stepsize

Change in DT pole: problem specific or property of forward Euler?

Approach: make a systems model of forward Euler method.

CT block diagrams: adders, gains, and integrators:

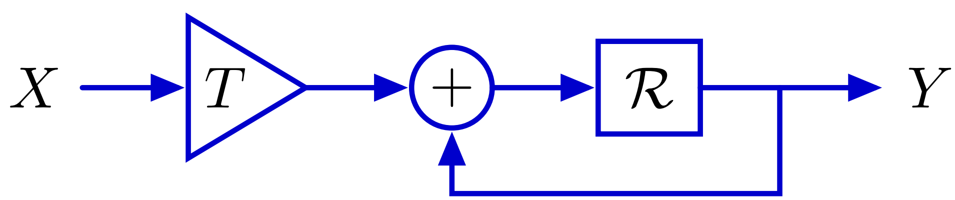


$$\dot{y}(t) = x(t)$$

Forward Euler approximation:

$$\frac{y[n+1] - y[n]}{T} = x[n]$$

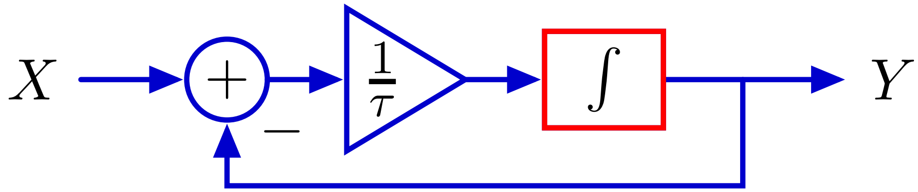
Equivalent system:



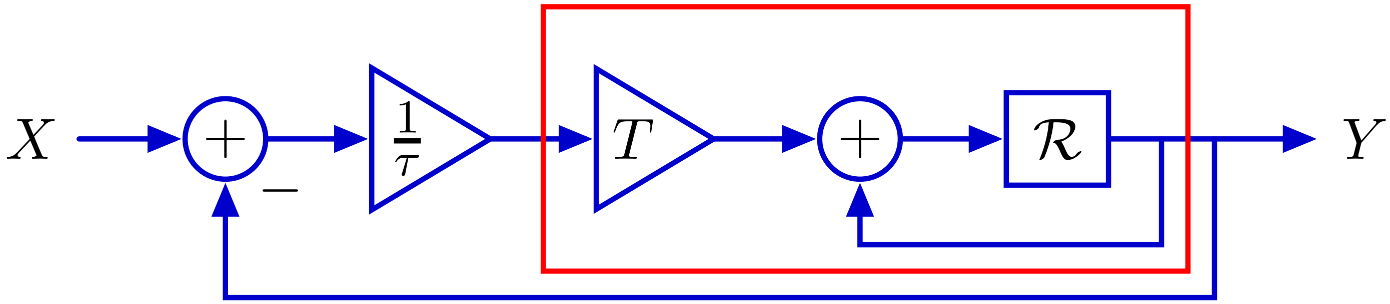
Forward Euler: substitute equivalent system for all integrators.

Example: leaky tank system

Started with leaky tank system:



Replace integrator with forward Euler rule:



Write system functional:

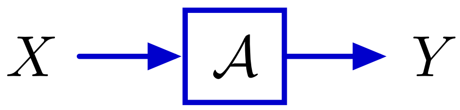
$$\frac{Y}{X} = \frac{\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}}{1+\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\mathcal{R}+\frac{T}{\tau}\mathcal{R}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\left(1-\frac{T}{\tau}\right)\mathcal{R}}$$

Equivalent to system we previously developed:

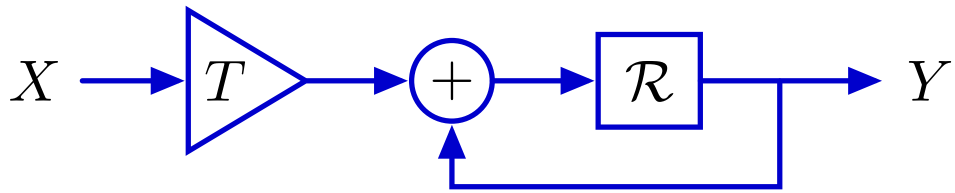
$$y_d[n+1] - \left(1-\frac{T}{\tau}\right)y_d[n] = \frac{T}{\tau}x_d[n]$$

Model of Forward Euler Method

Replace every integrator in the CT system



with the forward Euler model:



Substitute the DT operator for $\mathcal{A}$:

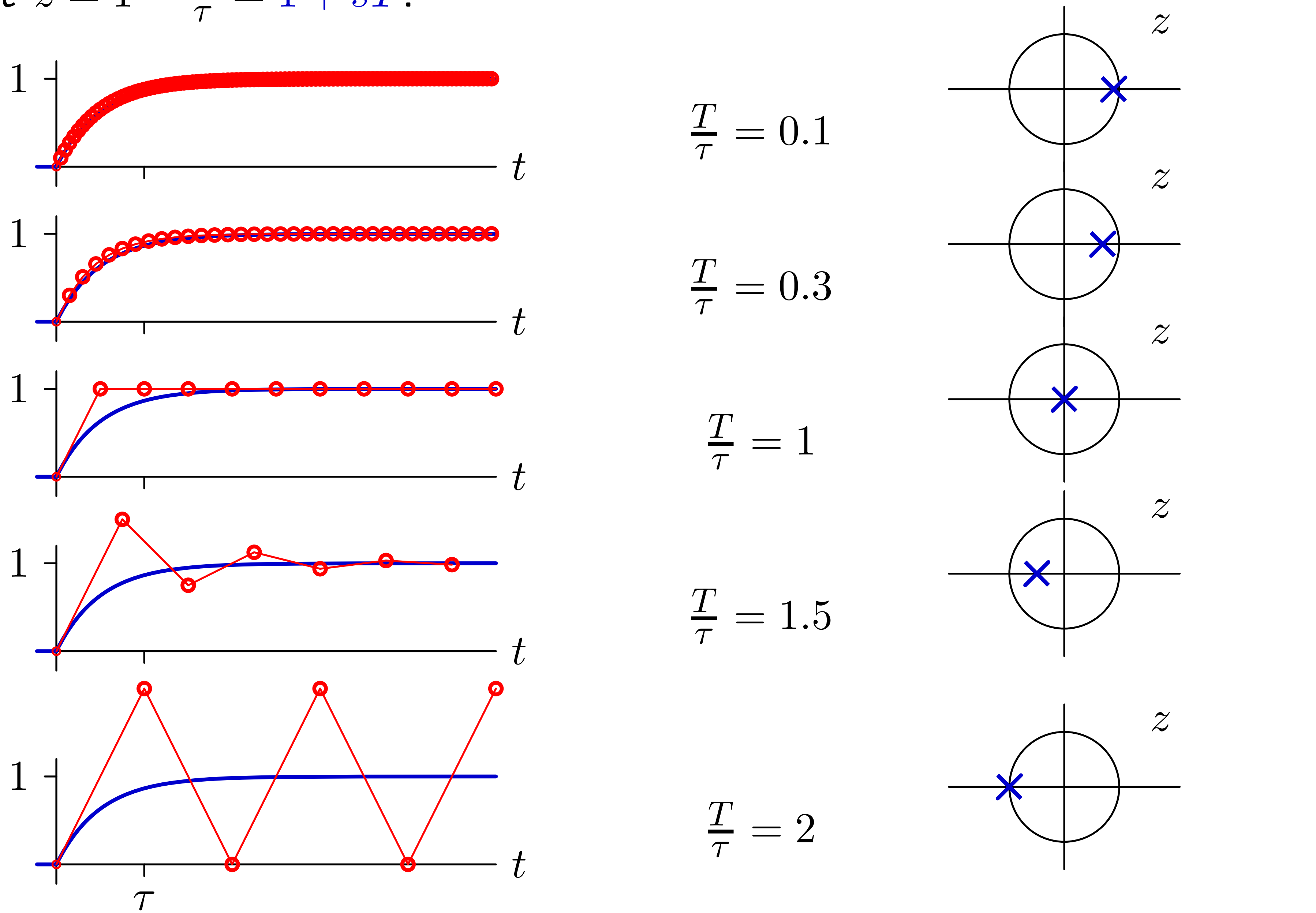
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T\mathcal{R}}{1-\mathcal{R}} = \frac{\frac{T}{z}}{1-\frac{1}{z}} = \frac{T}{z-1}$$

Forward Euler maps $s \rightarrow \dfrac{z-1}{T}$.

Or equivalently: $z = 1 + sT$.

Dependence of DT pole on Stepsize

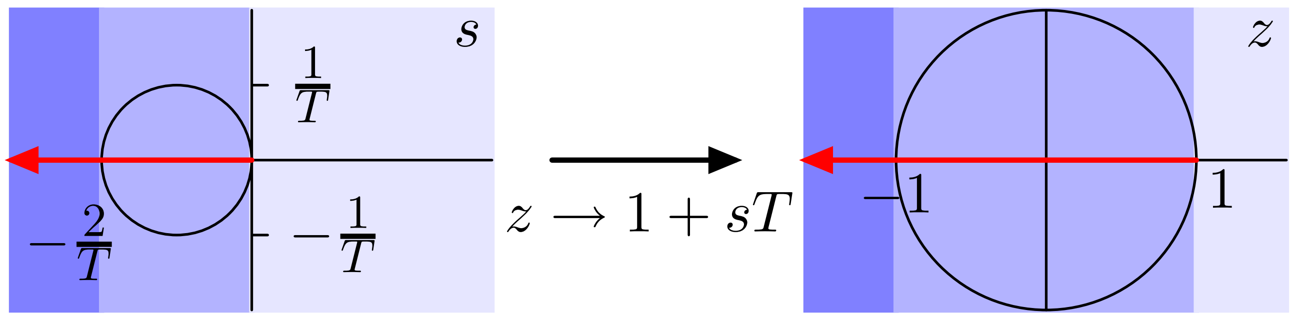
Pole at $z = 1 - \frac{T}{\tau} = 1 + sT$.



Forward Euler: Mapping CT poles to DT poles

Forward Euler Map:

s	$\longrightarrow$	$z = 1 + sT$
0		1
$-\frac{1}{T}$		0
$-\frac{2}{T}$		-1



DT stability: CT pole must be inside circle of radius $\frac{1}{T}$ at $s = -\frac{1}{T}$.

$$-\frac{2}{T} < -\frac{1}{\tau} < 0 \qquad \rightarrow \qquad \frac{T}{\tau} < 2$$

Backward Euler Approximation

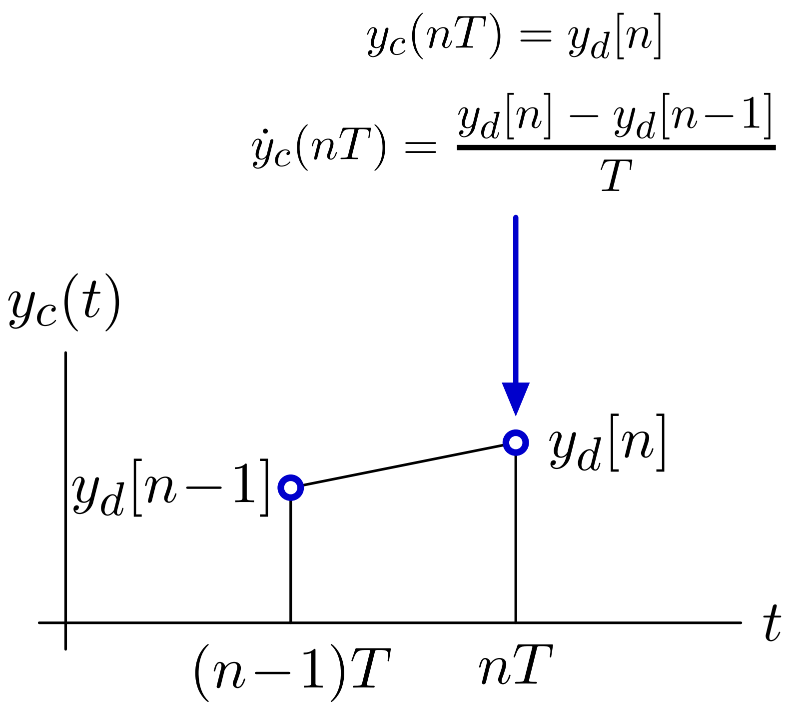
We can do a similar analysis of the backward Euler method.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big(nT\big) - y_c\big((n-1)T\big)}{T} = \frac{y_d[n] - y_d[n-1]}{T}$$



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into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

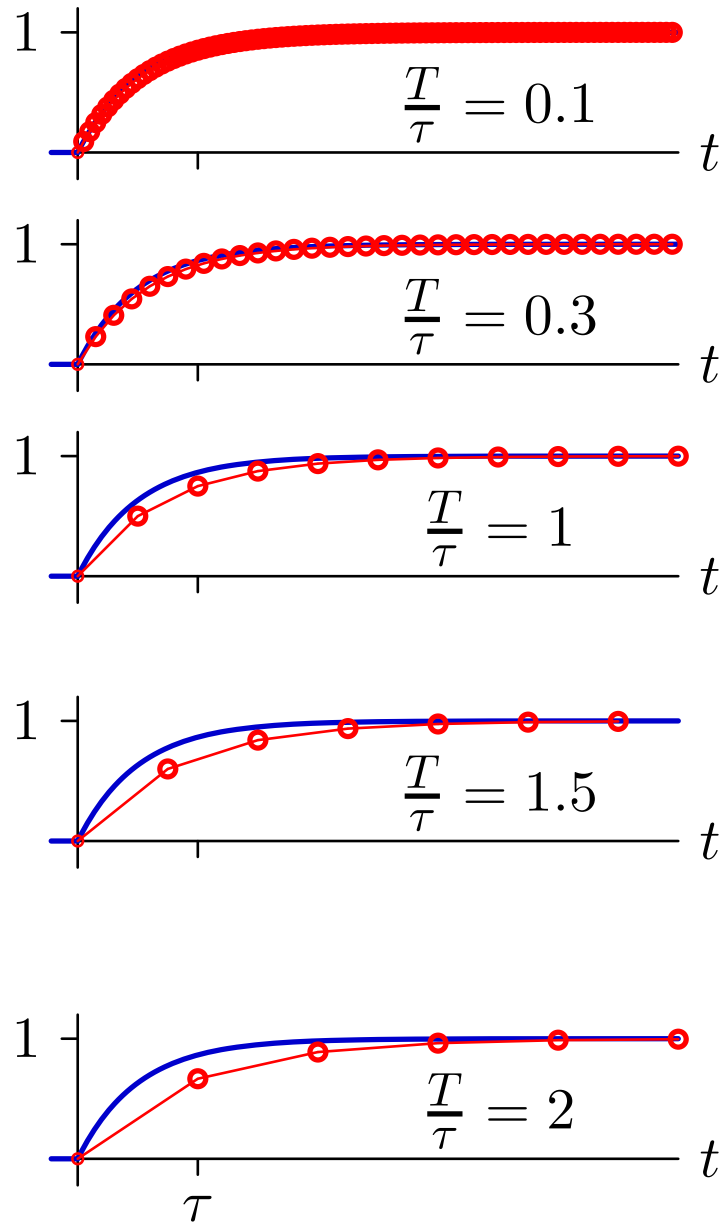
$$\frac{\tau}{T}\Big(y_d[n] - y_d[n-1]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$\left(1 + \frac{T}{\tau}\right) y_d[n] - y_d[n-1] = \frac{T}{\tau} x_d[n]$$

Backward Euler Approximation

Plot.



This approximation is better behaved. Why?

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Find the DT pole.

1. $z=\frac{T}{\tau}$
2. $z=1-\frac{T}{\tau}$
3. $z=\frac{\tau}{T}$
4. $z=-\frac{\tau}{T}$
5. $z=\frac{1}{1+\frac{T}{\tau}}$

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Take the Z transform:

$$\left(1+\frac{T}{\tau}\right)Y_d(z)-z^{-1}Y_d(z)=\frac{T}{\tau}X_d(z)$$

Find the system function:

$$H(z)=\frac{Y_d(z)}{X_d(z)}=\frac{\frac{T}{\tau}z}{\left(1+\frac{T}{\tau}\right)z-1}$$

Pole at $z=\dfrac{1}{1+\frac{T}{\tau}}$.

Check Yourself

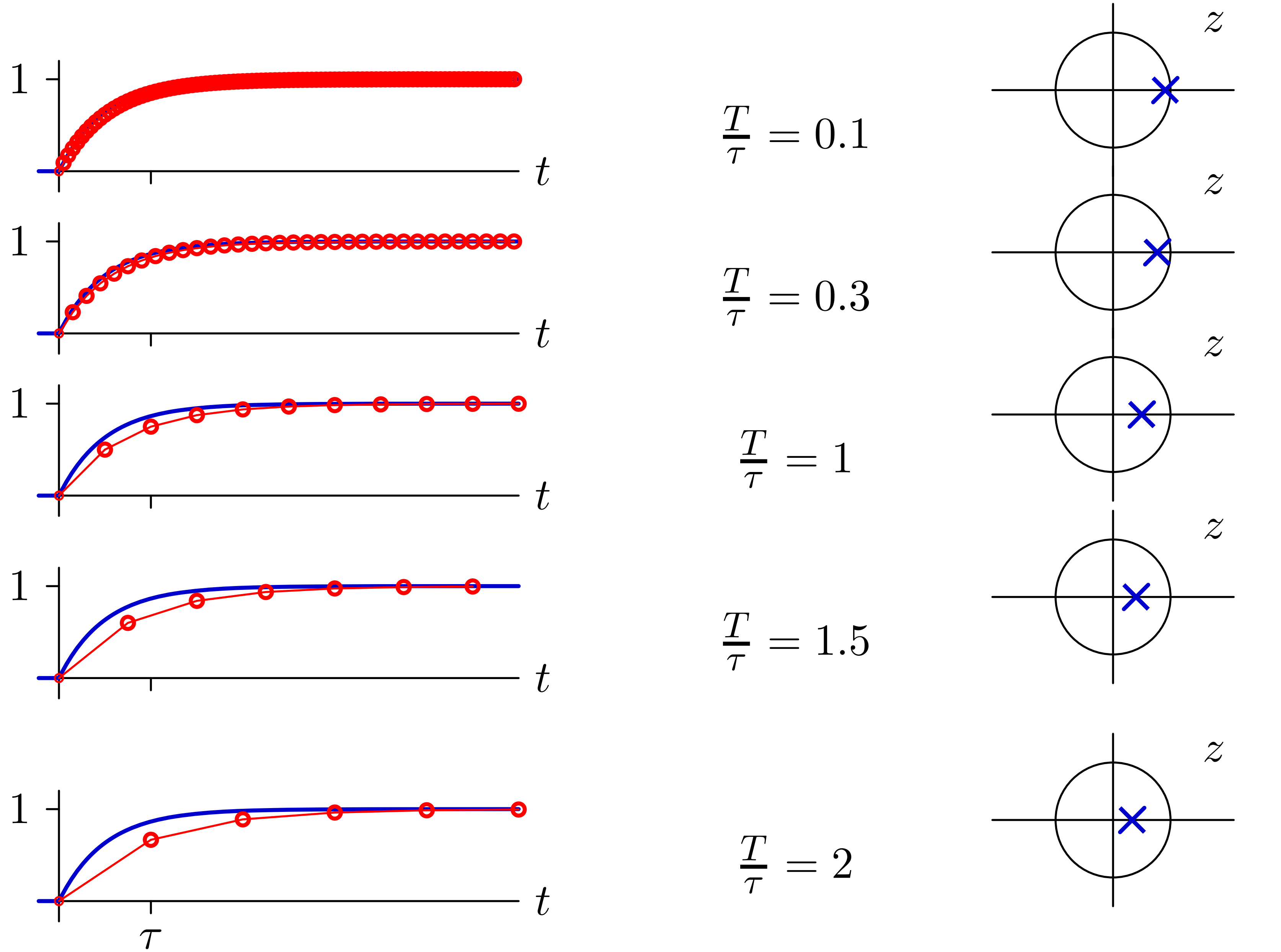
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 5

1. $z = \dfrac{T}{\tau}$
2. $z = 1 - \dfrac{T}{\tau}$
3. $z = \dfrac{\tau}{T}$
4. $z = -\dfrac{\tau}{T}$
5. $z = \dfrac{1}{1 + \frac{T}{\tau}}$

Dependence of DT pole on Stepsize

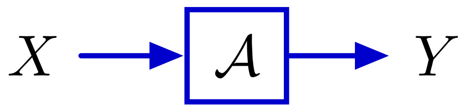


Why is this approximation better behaved?

Dependence of DT pole on Stepsize

Make a systems model of backward Euler method.

CT block diagrams: adders, gains, and integrators:

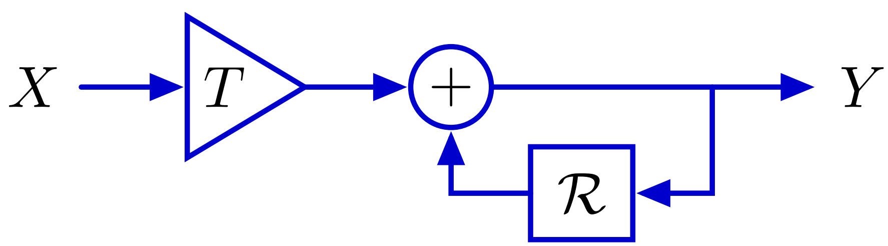


$$\dot{y}(t) = x(t)$$

Backward Euler approximation:

$$\frac{y[n] - y[n-1]}{T} = x[n]$$

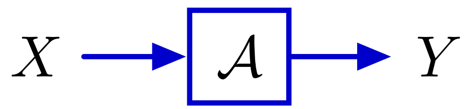
Equivalent system:



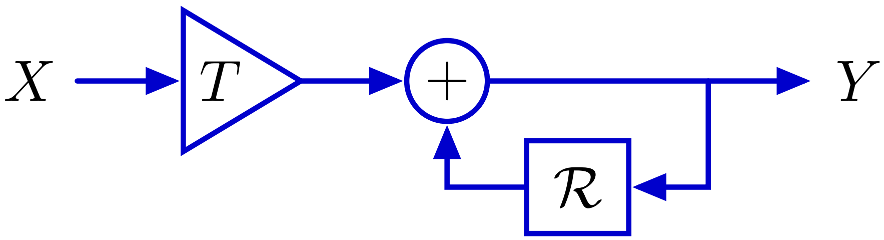
Backward Euler: substitute equivalent system for all integrators.

Model of Backward Euler Method

Replace every integrator in the CT system



with the backward Euler model:



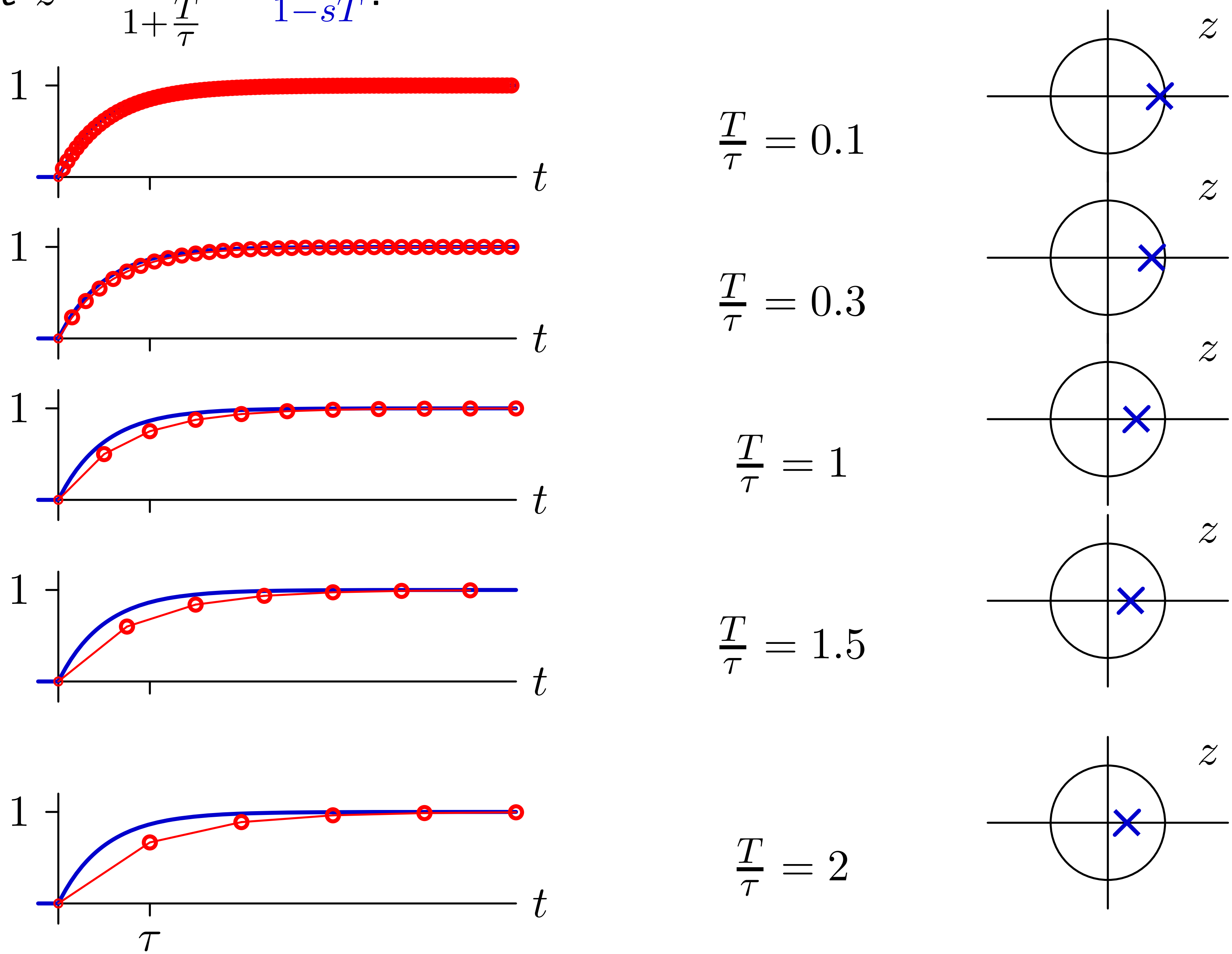
Substitute the DT operator for $\mathcal{A}$:

$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{1-\mathcal{R}} = \frac{T}{1-\frac{1}{z}}$$

Backward Euler maps $z \rightarrow \dfrac{1}{1-sT}$.

Dependence of DT pole on Stepsize

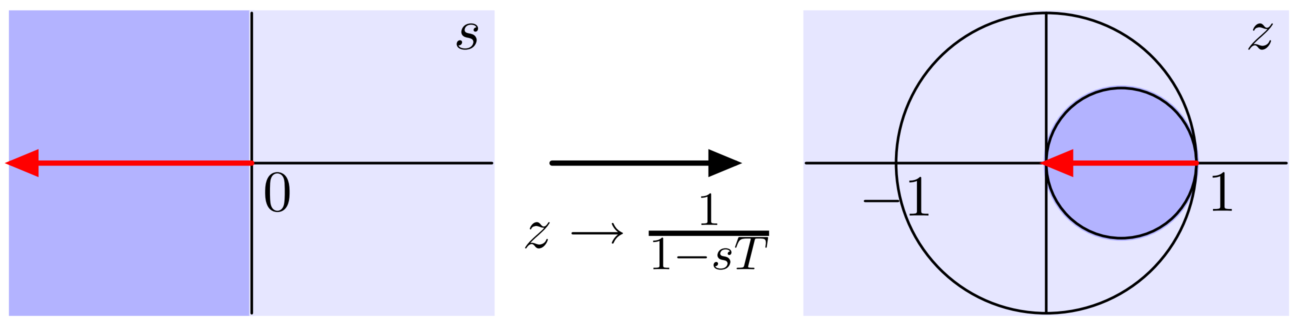
Pole at $z = \frac{1}{1+\frac{T}{\tau}} = \frac{1}{1-sT}$.



Backward Euler: Mapping CT poles to DT poles

Backward Euler Map:

s	$\rightarrow$	$z = \frac{1}{1-sT}$
0		1
$-\frac{1}{T}$		$\frac{1}{2}$
$-\frac{2}{T}$		$\frac{1}{3}$



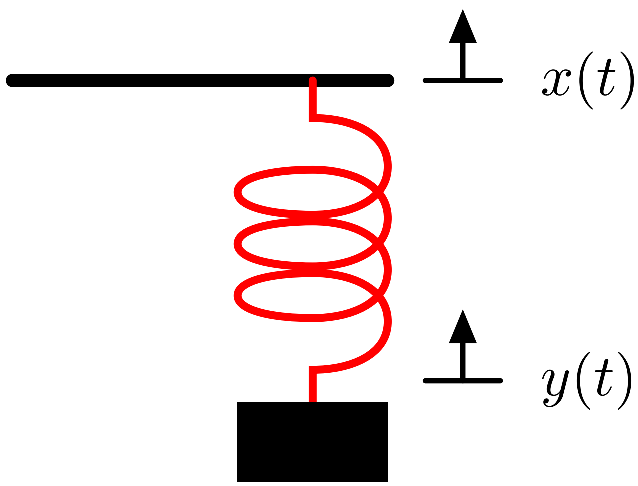
The entire left half-plane maps inside a circle with radius $\frac{1}{2}$ at $z = \frac{1}{2}$.

If CT system is stable, then DT system is also stable.

Masses and Springs, Forwards and Backwards

In Homework 2, you investigated three numerical approximations to a mass and spring system:

- forward Euler
- backward Euler
- centered method



Trapezoidal Rule

The trapezoidal rule uses centered differences.

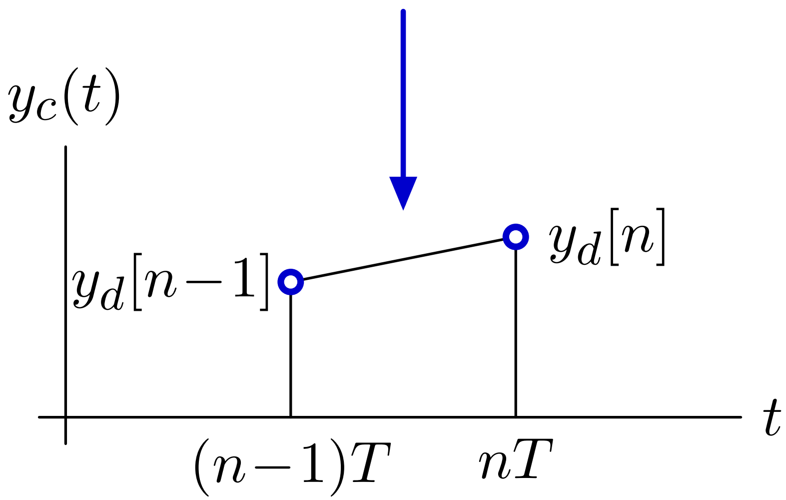
$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

$$y_c\Big(\big(n+\frac{1}{2}\big)T\Big) = \frac{y_d[n] + y_d[n-1]}{2}$$

$$\dot{y}_c\Big(\big(n+\frac{1}{2}\big)T\Big) = \frac{y_d[n] - y_d[n-1]}{T}$$



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Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

Z transform:

$$H(z) = \frac{Y(s)}{X(s)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Map:

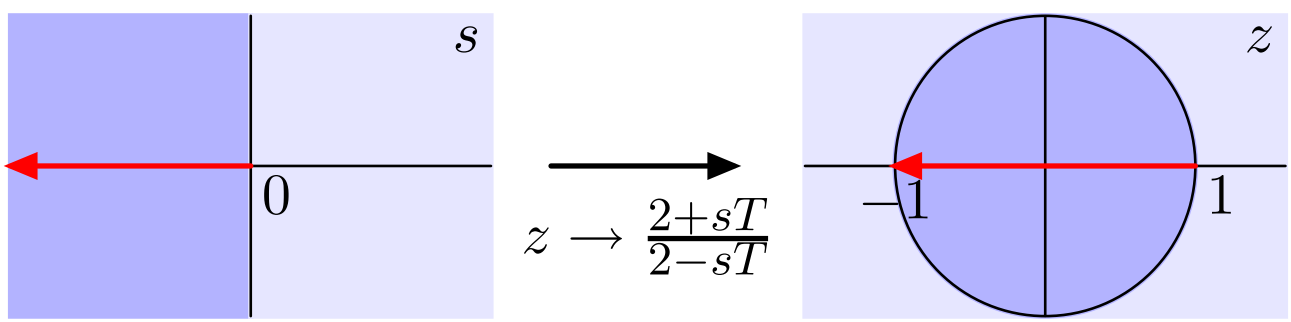
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Trapezoidal rule maps $z \rightarrow \dfrac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$.

Trapezoidal Rule: Mapping CT poles to DT poles

Trapezoidal Map:

s	$\longrightarrow$	$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$
0		1
$-\frac{1}{T}$		$\frac{1}{3}$
$-\frac{2}{T}$		0
$-\infty$		-1
$j\omega$		$\frac{2+j\omega T}{2-j\omega T}$

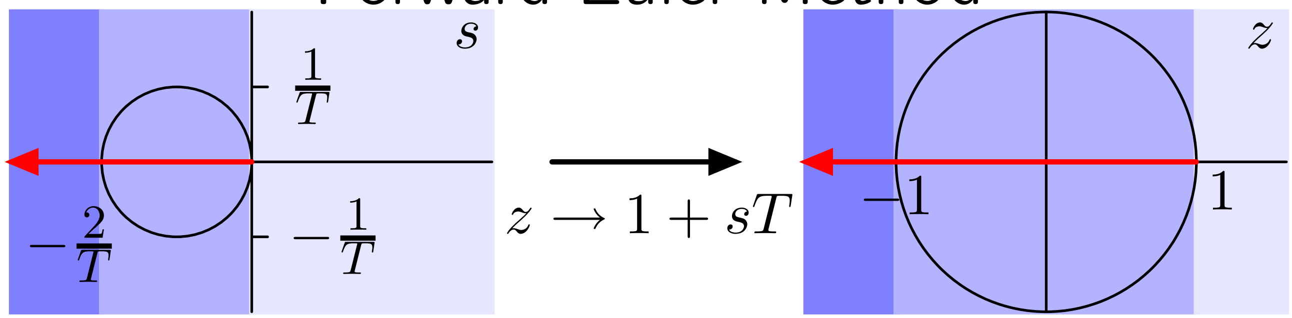


The entire left-half plane maps inside the unit circle.

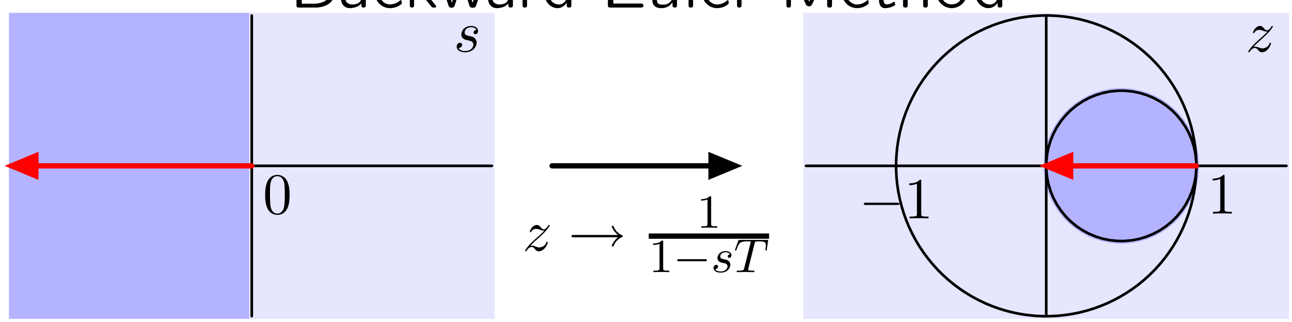
The $j\omega$ axis maps onto the unit circle

Mapping s to z: Leaky-Tank System

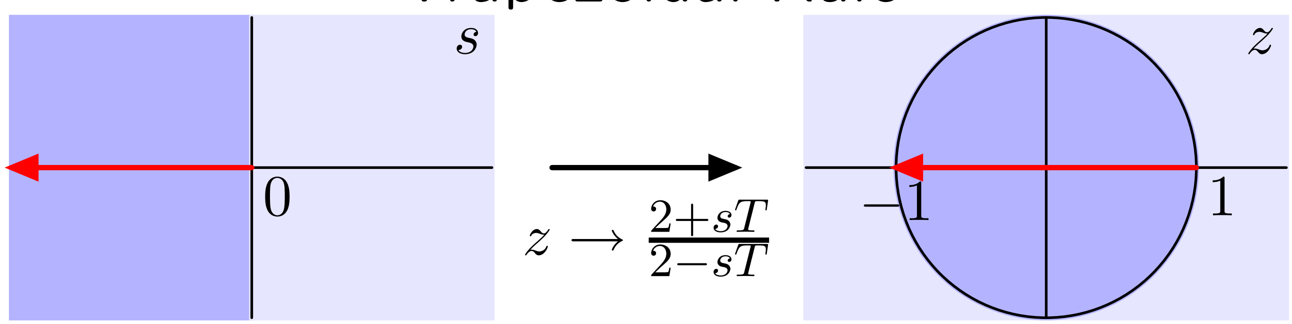
Forward Euler Method



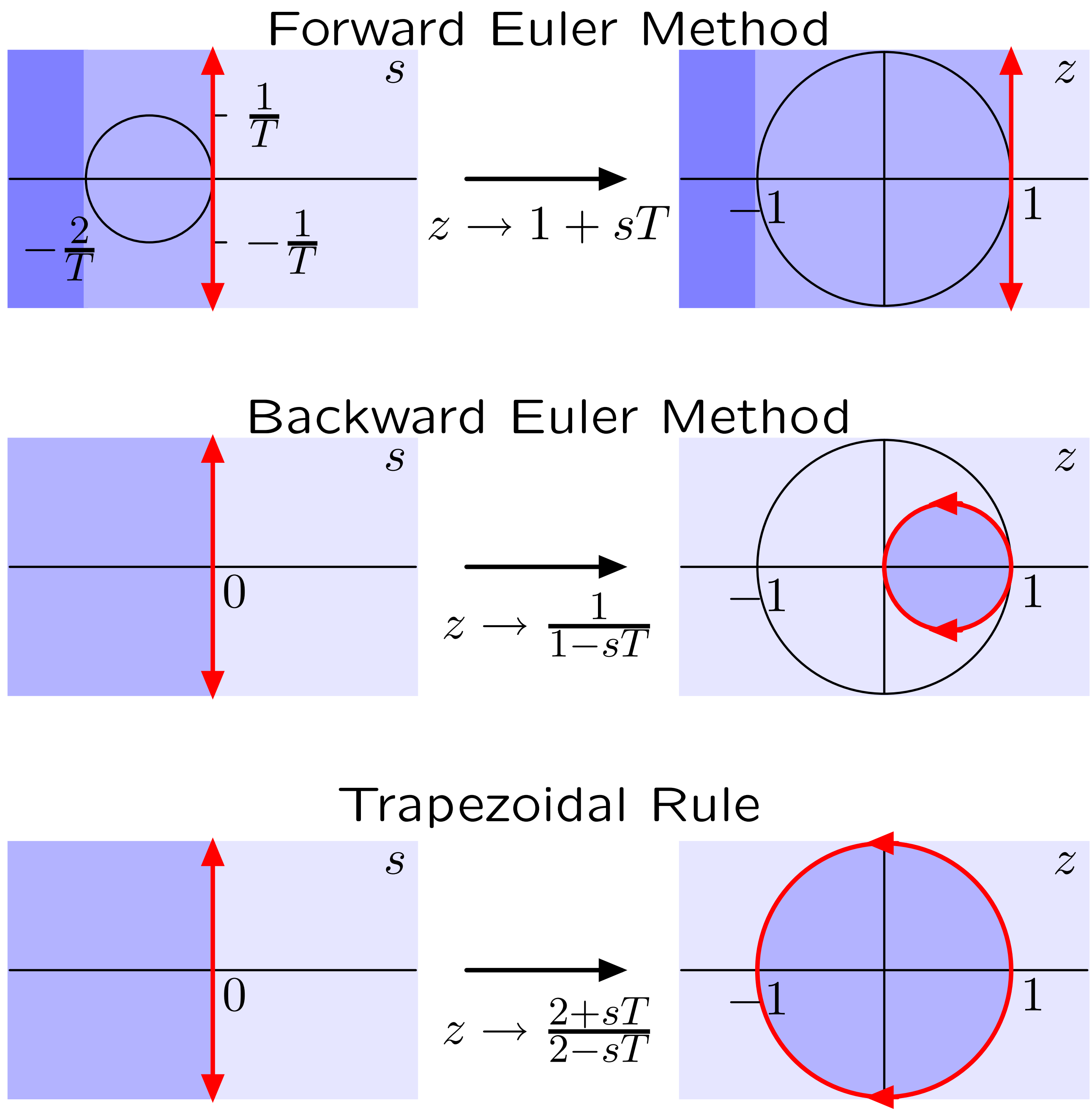
Backward Euler Method



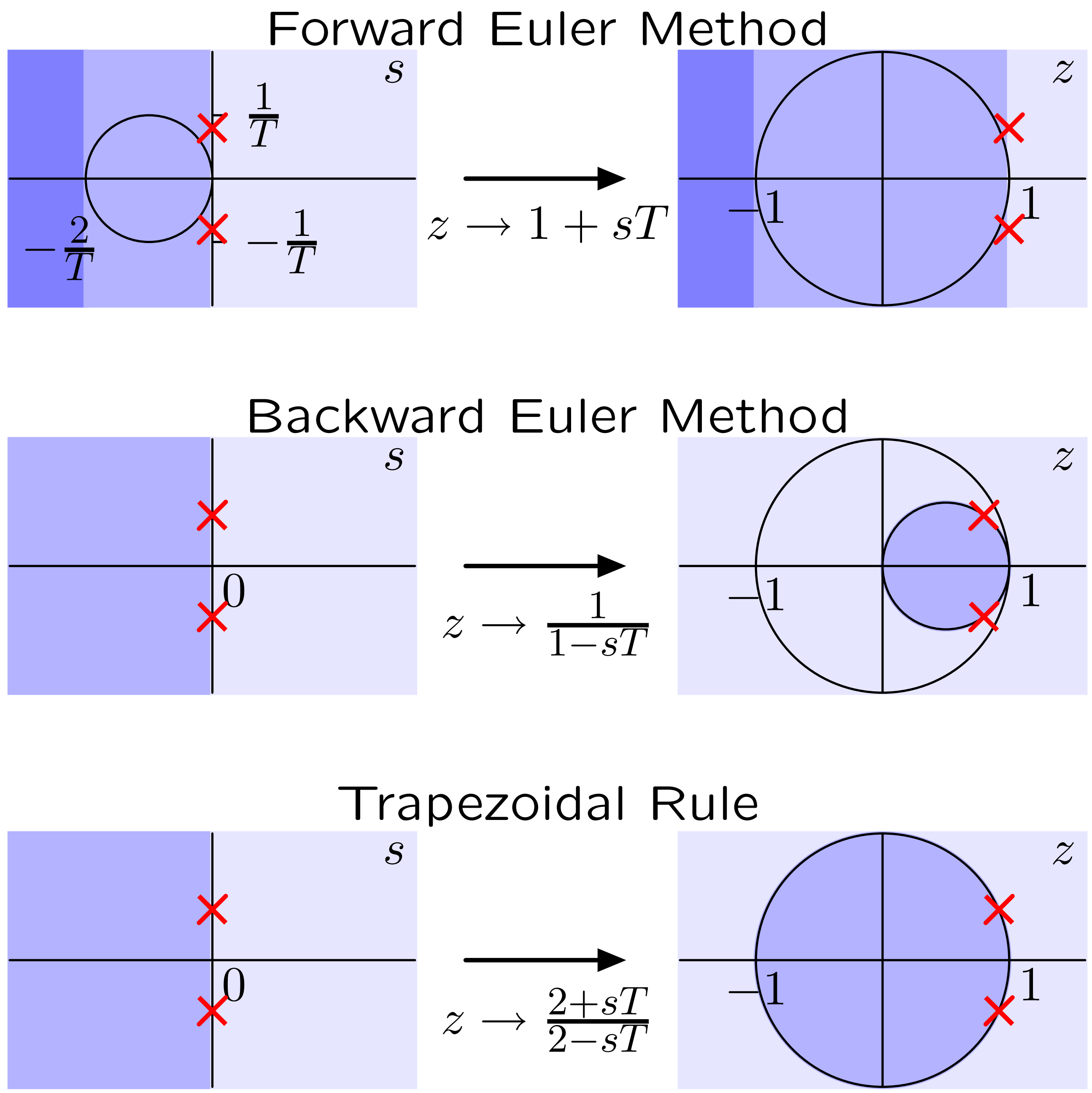
Trapezoidal Rule



Mapping s to z: Mass and Spring System

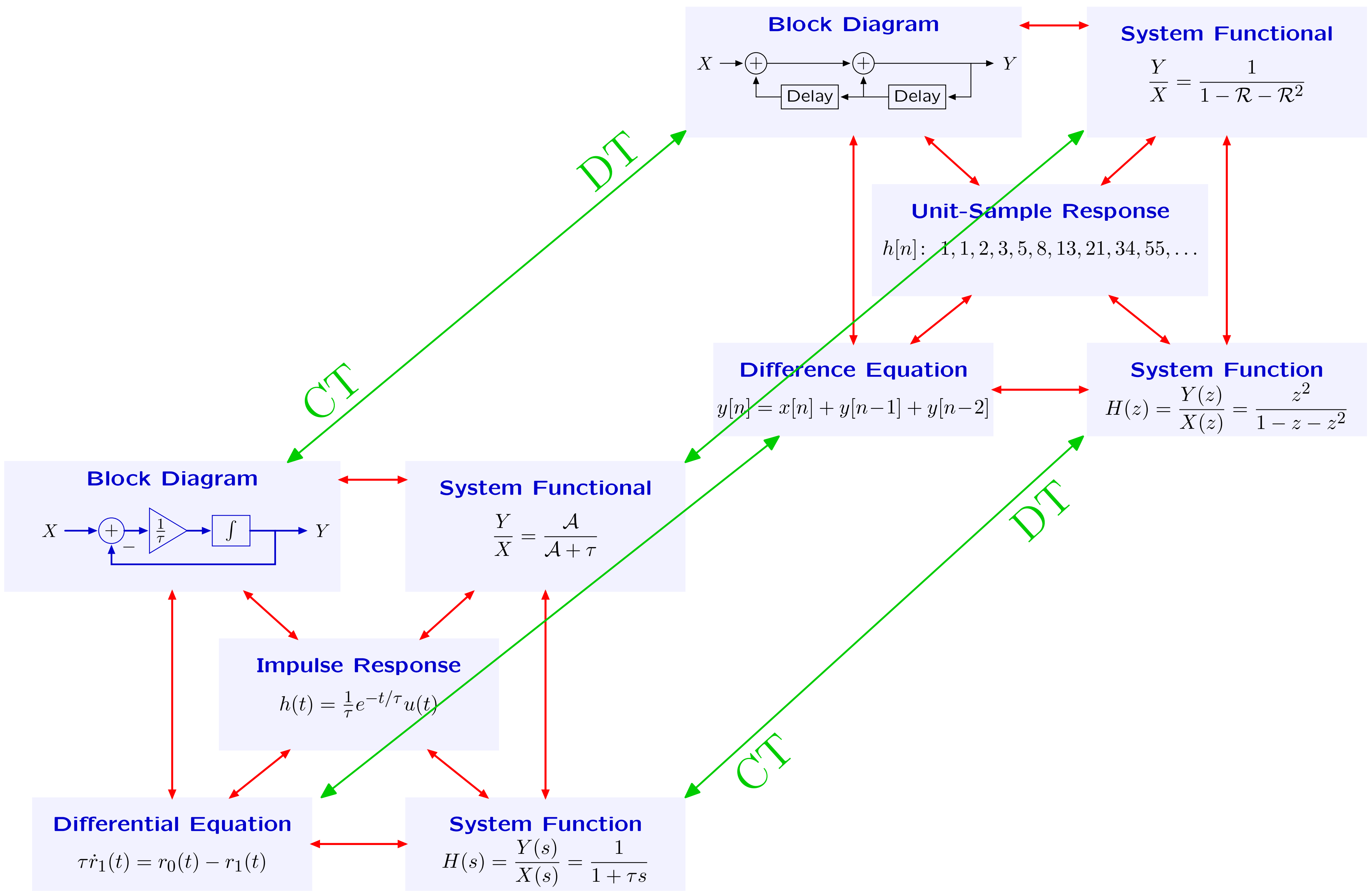


Mapping s to z: Mass and Spring System



Concept Map

Relations between CT and DT representations.



6.003 Signals and Systems
Spring 2010