# Quiz II Review
# Signals and Systems
# 6.003

Massachusetts Institute of Technology

April 5, 2010

# Quiz 2 Details

- *Date:* Wednesday April 7th, 2010
- *Time:* 7.30pm–9.30pm
- *Content:* (boundaries inclusive)
  - Lectures 1–15
  - Recitations 1–15
  - Homeworks 1–8

# Review Outline

- CT and DT system representations
- The impulse response
- Convolution
- Eigenfunctions
- Frequency response
- Complex numbers and vector methods
- Bode plots
- Feedback and Control
- Fourier series

# CT and DT System Representations

- Verbal descriptions: The answer to "what does this system do to a signal at its input?"
- Difference/differential equations: Mathematically compact and show the explicit mathematical relationship between input/output.

$$y[n] = x[n] + z_0 y[n-1]$$

- Block diagrams: Illustrate signal flow paths
- Operator representations: Analyze systems as polynomials.

$$\frac{Y}{X} = \frac{1}{1 - z_0 R}$$

- Transforms: Represents differential/difference equations with algebraic equations

$$H(z) = \frac{z}{z - z_0}$$

- LTI systems: representation by the **impulse response**

# The Impulse Response

*Why does this make sense?*

- Because every signal can be thought of in terms of a sum (or superposition) of shifted and scaled unit samples.
- From **Linearity** of the system:

$$x[n] = c_1 x_1[n] + c_2 x_2[n] + c_3 x_3[n]$$
$$y[n] = c_1 y_1[n] + c_2 y_2[n] + c_3 y_3[n]$$

- From **Time Invariance** of the system:

$$x[n] = x_1[n - n_0]$$
$$y[n] = y_1[n - n_0]$$

- Putting these together:

$$x[n] = -2\delta[n+1] + 2\delta[n] + \delta[n-1]$$
$$y[n] = -2h[n+1] + 2h[n] + h[n-1]$$

# Convolution

In previous example -2, 2, and 1 multiplying the shifted responses to $\delta[n]$ are actually values of x[k] for k=-1,0,1

In general:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

This is known as **convolution**

$$y[n] = x[n] * h[n] = (x * h)[n]$$

# Convolution (cont)

Some important concepts from convolution:

- Graphical interpretation: flip, shift, multiply, and sum!
- Convolution with an impulse

$$x(t) * \delta(t) = x(t)$$

- Convolution with a shifted impulse

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

- The Laplace/Z transform of convolution is multiplication

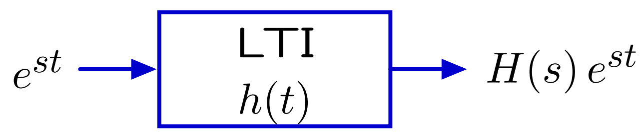
$$y(t) = x(t) * h(t)$$
$$Y(s) = X(s)H(s)$$

- Review HW5 Q3

# Eigenfunctions

When an eigenfunction is used as an input, the output is just a scaled version of the input.

<u>Claim</u>: Eternal exponentials are eigenfunctions of LTI systems:




<u>Proof</u>:

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s_0(t-\tau)} d\tau$$

$$= e^{s_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-s_0 \tau} d\tau$$

$$= e^{s_0 t} H(s)|_{s=s_0} = e^{s_0 t} H(s_0)$$

# Frequency Response

$$H(s)|_{s=j\omega}$$

*Why do we call this the frequency response?*

Example: If the input to a system is a cosine, what is the output?

$$x(t) = cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$\begin{aligned}
y(t) &= \frac{1}{2}H(j\omega_0)e^{j\omega_0 t} + \frac{1}{2}H(-j\omega_0)e^{-j\omega_0 t} \\
&= Re\{H(j\omega_0)e^{j\omega_0 t}\} = Re\{|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\} \\
&= |H(j\omega_0)|Re\{e^{j(\omega_0 t + \angle H(j\omega_0))}\} \\
&= |H(j\omega_0)|cos(\omega_0 t + \angle H(j\omega_0))
\end{aligned}$$

# Complex Numbers and Vector Methods

We can think of complex numbers as vectors in the complex plane with magnitude $r$ and angle $\theta$.

$$a + jb = \sqrt{a^2 + b^2} e^{j \arctan\left(\frac{b}{a}\right)} = re^{j\theta}$$

This can help us graphically compute the frequency response.
For example:

$$\begin{aligned} H(j\omega) &= \frac{(j\omega + \alpha_1)(j\omega + \alpha_2)}{(j\omega + \alpha_3)(j\omega + \alpha_4)} \\ &= \frac{r_1 e^{j\theta_1} r_2 e^{j\theta_2}}{r_3 e^{j\theta_3} r_4 e^{j\theta_4}} \\ &= \frac{r_1 r_2}{r_3 r_4} e^{j(\theta_1 + \theta_2 - \theta_3 - \theta_4)} \end{aligned}$$

$$|H(j\omega)| = \frac{r_1 r_2}{r_3 r_4}$$

$$\angle H(j\omega) = \theta_1 + \theta_2 - \theta_3 - \theta_4$$

# Bode Plots

- The **amplitude** of $H(j\omega)$ tells us how much an incoming cosine with frequency $\omega$ will be **scaled**.
- The **phase** of $H(j\omega)$ tells us how much an incoming cosine with frequency $\omega$ will **shift its argument** (angle).

Thus, we would like to know how amplitude and phase of $H(j\omega)$ change with $\omega$. These plots are called Bode Plots.

The magnitude is typically in dB, plotted against $\omega$ in the log scale.

$$|H_{dB}(j\omega)| = 20\log|H(j\omega)|$$

# Bode Plots (cont)

Bode plots are most easily constructed using isolated building blocks.

$$\log H_1(s)H_2(s)H_3(s) = \log H_1(s) + \log H_2(s) + \log H_3(s)$$

For real poles and zeros use these first order building blocks:

$$H(s) = s,\ H(s) = \frac{1}{s},\ H(s) = s + a,\ H(s) = \frac{1}{s+a}$$

If the system has complex poles use 2nd order building blocks:

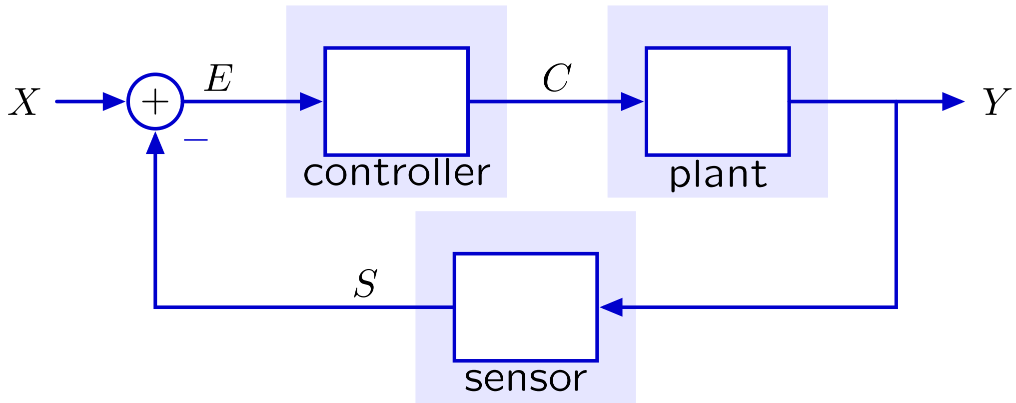
$$H(s) = \frac{A}{1 + \frac{1}{Q}\frac{s}{\omega_0} + (\frac{s}{\omega_0})^2}$$

Complex poles of **real systems** will come in complex conjugate pairs.

$$\frac{s}{\omega_0} = -\frac{1}{2Q} \pm j\sqrt{1 - \left(\frac{1}{2Q}\right)^2}$$

*High Q means slow decay rate and fast oscillations*

# Feedback and Control




- Black's Equation!

$$H(s) = \text{feed through transmission}/(1 - \textit{looptransmission})$$

- Stability (for causal systems): $Re(s) < 0$ for CT, $|z| < 1$ for DT
- Stability (in general): $j\omega$ axis (unit circle for DT) in the ROC
- A system is said to be BIBO stable if any bounded input produces a bounded output. A system has BIBO stability if the impulse response is *absolutely integrable (summable)* (HW7 Q3)

# Fourier Series

- Periodic signals can be represented by a sum of harmonics
- The integral over one period of a harmonic is equal to zero, except for k=0.

$$\int_T e^{jk\omega_0 t} dt = T\delta[k]$$

- The "analysis" equation gives us the Fourier coefficients

$$a_k = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$$

- The "synthesis" equation reconstructs the periodic signal

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

- This representation allows us to think of systems as **filters**

# End of Review

Good luck on Wednesday! :-)

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