6.453 Quantum Optical Communication
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6.453 Quantum Optical Communication

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Reading: For classical coupled-mode equations for parametric interactions:

- B.E.A. Saleh and M.C. Teich, *Fundamentals of Photonics,* (Wiley, New York, 1991) section 19.4.

Introduction

The final major question we shall address this semester is the following. How can we create non-classical light beams that exhibit the signatures we've discussed in our simple one-mode and two-mode analyses? In particular, we will study spontaneous parametric downconversion and optical parametric amplification in second-order nonlinear crystals. These closely-related processes have been and continue to be the primary vehicles for generating non-classical light beams. Given our interest in the system-theoretic aspects of quantum optical communication—and our lack of a serious electromagnetic fields prerequesite—we shall tread lightly, focusing on the coupled-mode equations characterization of collinear configurations, i.e., we shall suppress transverse spatial effects. Nevertheless, we will be able to get to the basic physics of these interactions and provide continuous-time versions of the non-classical signatures that we discussed in single-mode and two-mode forms earlier this term. Today, however, we will begin with a treatment within the classical domain. In the two lectures to follow we will convert today's material into the quantum domain, and then explore the implications of that quantum characterization.

Spontaneous Parametric Downconversion

Slide 3 shows a conceptual picture of spontaneous parametric downconversion (SPDC). A strong laser-beam pump is applied to the entrance facet (at $z = 0$) of a crystalline material that possesses a second-order ($\chi^{(2)}$) nonlinearity. We will only concern ourselves with continuous-wave (cw) pump fields, so this pump beam will be taken to be monochromatic at frequency ω_P. Even though the only light applied to the crystal is at frequency ω_P, three-wave mixing in this nonlinear material can result in the production of lower-frequency signal and idler waves, with center frequencies ω_S

and ω_I, respectively, that emerge—along with the transmitted pump beam—from the crystal's output facet (at $z = l$). This process is *downconversion*, because the signal and idler light arises from a higher-frequency pump beam. The process is deemed *parametric*, because the downconversion is due to the presence of the pump modifying the effective material parameters encountered by the fields propagating at the signal and idler frequencies. It is called *spontaneous*, because there is no illumination of the crystal's input facet at the signal and idler frequencies. Of course, this zero-field input statement is correct in a classical physics description of slide 3. We know, from our quantum description of the electromagnetic field, that the positive-frequency field operator at the crystal's input facet must include components at both the signal and idler frequencies. In SPDC, the $z = 0$ signal and idler frequencies are unexcited, i.e., in their vacuum states. The action of the pump beam in conjunction with the crystal's nonlinearity is responsible for the excitation at these frequencies that is seen at $z = l$. Thus, although a quantum analysis will be required to understand the SPDC process, we will devote the rest of today's effort to a classical treatment of the slide 3 configuration. Nevertheless, we shall get a hint of the quantum future because the signal and idler frequencies, in the classical theory, will obey $\omega_S + \omega_I = \omega_P$. Zero-valued input fields at the signal and idler frequencies cannot account for the energy in non-zero signal and idler output fields. Instead, the energy present in these output fields must come from the pump beam. Rewriting the preceding frequency condition as $\hbar\omega_S + \hbar\omega_I = \hbar\omega_P$ at least *suggests* that a photon fission process—in which a single pump photon spontaneously downconverts into a signal photon plus an idler photon such that energy is conserved—is what is happening in SPDC. In fact, such is the case.

Maxwell's Equations in a Nonlinear Dielectric Medium

We will start our classical analysis of electromagnetic wave propagation in a $\chi^{(2)}$ medium from bedrock: Maxwell's equations for propagation in a source-free region of a nonlinear dielectric. In differential form, and *without* assuming any constitutive laws, we have that

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t), \quad \text{Faraday's law} \tag{1}$$

$$\nabla \cdot \vec{D}(\vec{r},t) = 0, \quad \text{Gauss' law} \tag{2}$$

$$\nabla \times \vec{H}(\vec{r},t) = \frac{\partial}{\partial t}\vec{D}(\vec{r},t), \quad \text{Ampère's law} \tag{3}$$

$$\nabla \cdot \vec{B}(\vec{r},t) = 0, \quad \text{Gauss' law for the magnetic flux density,} \tag{4}$$

where $\vec{E}(\vec{r},t)$ is the electric field, $\vec{D}(\vec{r},t)$ is the displacement flux density, $\vec{H}(\vec{r},t)$ is the magnetic field, and $\vec{B}(\vec{r},t)$ is the magnetic flux density. All of these fields are real valued and in SI units. For dielectrics, we can take

$$\vec{B} = \mu_0 \vec{H}(\vec{r}, t), \tag{5}$$

where μ_0 is the permeability of free space, as one of the material's constitutive laws. The other free-space constitutive law is

$$\vec{D}(\vec{r}, t) = \epsilon_0 \vec{E}(\vec{r}, t), \tag{6}$$

where ϵ_0 is the permittivity of free space.[1] However, for the nonlinear dielectric of interest here we will use

$$\vec{D}(\vec{r}, t) = \epsilon_0 \vec{E}(\vec{r}, t) + \vec{P}(\vec{r}, t), \tag{7}$$

where $\vec{P}(\vec{r}, t)$ is the material's polarization, which is a nonlinear function of the electric field.

Our initial objective is to reduce Maxwell's equations to a wave equation for a $+z$-propagating plane wave. Taking the curl of Faraday's law, employing the vector identity

$$\nabla \times [\nabla \times \vec{F}(\vec{r}, t)] = \nabla[\nabla \cdot \vec{F}(\vec{r}, t)] - \nabla^2 \vec{F}(\vec{r}, t), \tag{8}$$

and Ampère's law, we get

$$\nabla[\nabla \cdot \vec{E}(\vec{r}, t)] - \nabla^2 \vec{E}(\vec{r}, t) = -\mu_0 \frac{\partial}{\partial t}[\nabla \times \vec{H}(\vec{r}, t)] = -\mu_0 \frac{\partial^2}{\partial t^2} \vec{D}(\vec{r}, t). \tag{9}$$

For a $+z$-propagating plane wave whose electric field is orthogonal to the z axis, the preceding result simplifies to

$$\frac{\partial^2}{\partial z^2} \vec{E}(z, t) - \mu_0 \frac{\partial^2}{\partial t^2} \vec{D}(z, t) = \vec{0}. \tag{10}$$

Before moving on to propagation in the nonlinear medium, let's examine the wave solutions to Eq. (10) in free space and in a linear dielectric. Using $\vec{D}(z, t) = \epsilon_0 \vec{E}(\zeta, t)$, for free space, Eq. (10) becomes

$$\frac{\partial^2}{\partial z^2} \vec{E}(z, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E}(z, t) = \vec{0}, \tag{11}$$

where we have used $c = 1/\sqrt{\epsilon_0 \mu_0}$. It easily verified—recall Lecture 17—that

$$\vec{E}(z, t) = f(t - z/c) \vec{i}_f, \tag{12}$$

[1] In terms of ϵ_0 and μ_0 we have that $c = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum, as shown in Lecture 17.

is a solution to Eq. (11) for an arbitrary time function $f(t)$ and unit vector $\vec{i}_f$ in the x-y plane.[2] Moreover, this field is a $+z$-going plane wave, as was noted in Lecture 17.

Now suppose that we are interested in propagation through a linear dielectric. In this case, and for the nonlinear case to follow, it is best to go to the temporal-frequency domain, i.e., we define the Fourier transform of a field $\vec{F}(\vec{r}, t)$ by

$$\vec{\mathcal{F}}(\vec{r}, \omega) = \int \mathrm{d}t\, \vec{F}(\vec{r}, t) e^{j\omega t}. \tag{13}$$

The sign convention here is in keeping with our quantum-optics notion of what constitutes a positive-frequency field, viz., the inverse transform integral is

$$\vec{F}(\vec{r}, t) = \int \frac{\mathrm{d}\omega}{2\pi}\, \vec{\mathcal{F}}(\vec{r}, \omega) e^{-j\omega t}. \tag{14}$$

The constitutive law for a *linear* dielectric is

$$\vec{\mathcal{D}}(\vec{r}, \omega) = \epsilon_0[1 + \boldsymbol{\chi}^{(1)}(\omega)]\vec{E}(\vec{r}, \omega), \tag{15}$$

where the linear susceptibility, $\boldsymbol{\chi}^{(1)}(\omega)$, is a frequency-dependent tensor, so that the polarization,

$$\vec{\mathcal{P}}(\vec{r}, \omega) = \epsilon_0 \boldsymbol{\chi}^{(1)}(\omega)\vec{\mathcal{E}}(\vec{r}, \omega), \tag{16}$$

need *not* be parallel to the electric field. The tensor nature of the linear susceptibility is the anisotropy that we exploited in our discussion, earlier this semester, of wave plates. Thus, if $\vec{\mathcal{E}}(\vec{r}, \omega)$ is polarized along a principal axis of the crystal—as we shall assume in what follows—we have that

$$\vec{\mathcal{D}}(\vec{r}, \omega) = \epsilon_0 n^2(\omega)\vec{\mathcal{E}}(\vec{r}, \omega), \tag{17}$$

is the appropriate constitutive relation, where $n(\omega)$ is the refractive index at frequency ω for the chosen polarization. Now, if we take the Fourier transform of Eq. (10) and presume fields with no (x, y) dependence with an electric field polarized along a principal axis, we obtain the Helmholtz equation

$$\frac{\partial^2}{\partial z^2}\vec{\mathcal{E}}(z, \omega) + \frac{\omega^2 n^2(\omega)}{c^2}\vec{\mathcal{E}}(z, \omega) = \vec{0}. \tag{18}$$

The $+z$-going plane-wave solution to this equation is

$$\vec{\mathcal{E}}(z, \omega) = \mathrm{Re}[\vec{E} e^{-j(\omega t - kz)}]. \tag{19}$$

where $k \equiv \omega n(\omega)/c$ and $\vec{E}$ is a constant vector in the x-y plane.

[2] To show that Eq. (11) provides a solution to Maxwell's equations in free space, however, more work is needed. Faraday's law should be used to derive the associated magnetic field, $\vec{H}(z, t)$, and then it should be verified that $\vec{E}(z, t)$ *and* $\vec{H}(z, t)$ are solutions to the full set of Maxwell's equations. See Lecture 17 for more details.

For a *nonlinear* dielectric we shall employ the following frequency-domain constitutive relation:

$$\vec{\mathcal{D}}(\vec{r},\omega) = \epsilon_0 \boldsymbol{\chi}^{(1)}(\omega)\vec{\mathcal{E}}(\vec{r},\omega) + \vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega), \tag{20}$$

where $\boldsymbol{\chi}^{(1)}(\omega)$ is the medium's *linear* susceptibility tensor at frequency ω and $\vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$ is the *nonlinear* polarization, i.e., $\vec{\mathcal{P}}_{\rm NL}(\vec{r},\omega)$ is a nonlinear function of the electric field. Assuming, as before, a $+z$-going plane wave whose electric field is polarized along a principal axis of the $\boldsymbol{\chi}^{(1)}(\omega)$ tensor, Eq. (18) becomes

$$\frac{\partial^2}{\partial z^2}\vec{\mathcal{E}}(z,\omega) + \frac{\omega^2 n^2(\omega)}{c^2}\vec{\mathcal{E}}(z,\omega) = -\mu_0\omega^2\vec{\mathcal{P}}_{\rm NL}(z,\omega), \tag{21}$$

for the nonlinear dielectric. The left-hand side of this equation includes the medium's linear behavior, with its nonlinear character appearing as a source term on the right-hand side. General solutions to this equation—for arbitrary nonlinearities—are beyond our reach. In the next section, however, we show how to do a coupled-mode analysis that, when converted to quantum form in Lecture 21, will allow us to understand how SPDC produces non-classical light.

Coupled-Mode Equations

Here we shall delve deeper into propagation through a nonlinear dielectric when that material's nonlinear polarization arises from a second-order nonlinearity. Unlike the preceding section, which tried to work in generality, we will now assume that the electric field propagating from $z = 0$ to $z = L$ in the nonlinear crystal consists of three $+z$-going monochromatic plane waves: the frequency-ω_P pump beam; the frequency-ω_S signal beam; and the frequency-ω_I idler beam. Furthermore, we will assume that $\omega_P = \omega_S + \omega_I$ and that the pump is very strong while the signal and idler are very weak. Allowing—as will be necessary to account for the tensor properties of the second-order susceptibility—the pump, signal, and idler to have different linear polarizations along the crystal's principal axes, we will take the electric field to be

$$\begin{aligned}\vec{E}(z,t) &= \underbrace{{\rm Re}[A_S(z)e^{-j(\omega_S t - k_S z)}]\vec{i}_S}_{\text{signal}} + \underbrace{{\rm Re}[A_I(z)e^{-j(\omega_I t - k_I z)}]\vec{i}_I}_{\text{idler}} \\ &+ \underbrace{{\rm Re}[A_P e^{-j(\omega_P t - k_P z)}]\vec{i}_P}_{\text{pump}}, \quad \text{for } 0 \le z \le l.\end{aligned} \tag{22}$$

In this expression: $k_m = \omega_m n_m(\omega_m)/c$ for $m = S, I, P$ gives the wave numbers of the signal, idler, and pump fields in terms of the refractive indices, $n_m(\omega_m)$, of their respective linear polarizations, $\vec{i}_m$, which are all in the x-y plane. More importantly, for what will follow, the signal and idler complex envelopes, $A_S(z)$ and $A_I(z)$, are *slowly-varying* functions of z, i.e., they change very little on the scale of their field component's wavelength.[3] Also, the strong pump field has been taken to be non-depleting, i.e., its complex envelope, A_P, is a constant.[4] These assumptions are consistent with SPDC operation.

For the constitutive relation associated with the preceding electric field we will assume that

$$\begin{aligned}\vec{D}(z,t) &\approx \frac{\epsilon_0 n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t - k_S z)} + \text{cc}}{2}\vec{i}_S \\ &+ \frac{\epsilon_0 n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t - k_I z)} + \text{cc}}{2}\vec{i}_I \\ &+ \frac{\epsilon_0 n_P^2(\omega_P)A_P e^{-j(\omega_P t - k_P z)} + \text{cc}}{2}\vec{i}_P \\ &+ \frac{\epsilon_0 \chi^{(2)} A_I^*(z)A_P e^{-j[(\omega_P-\omega_I)t-(k_P-k_I)z]} + \text{cc}}{2}\vec{i}_S \\ &+ \frac{\epsilon_0 \chi^{(2)} A_S^*(z)A_P e^{-j[(\omega_P-\omega_S)t-(k_P-k_S)z]} + \text{cc}}{2}\vec{i}_I, \end{aligned} \quad (23)$$

where cc denotes complex conjugate. The first three terms on the right in Eq. (23) are due to the material's linear susceptibility. Except for the possibly different signal, idler, and pump polarizations, it is the three-wave version of what we exhibited in the previous section for a linear dielectric. The last two terms represent the effect of the material's second-order nonlinear susceptibility, $\chi^{(2)}$. Strictly speaking, this susceptibility is a frequency-dependent tensor that produces a nonlinear polarization $\vec{P}_{\text{NL}}(z,t)$ when it is multiplied by the product of two electric-field frequency components. In writing Eq. (23) we have suppressed the frequency dependence and tensor character by our choice of fixed frequencies and polarizations in Eq. (22), and we have only included second-order terms that appear at the signal or idler frequencies, as these are the frequencies that will be of interest in what follows, viz., they represent coupling between the signal and idler which is mediated by the presence of the strong pump beam in the nonlinear crystal.

Let us substitute Eq. (23) into Eq. (10) and exploit the linear independence of $e^{j\omega t}$ and $e^{-j\omega t}$ for $\omega \neq 0$ to restrict our attention to the positive-frequency terms. We

[3] This assumption goes by the acronym SVEA, i.e., the slowly-varying envelope approximation.

[4] Strictly speaking, this no-depletion assumption cannot be exactly correct, as the pump beam supplies the energy for the signal and idler outputs in SPDC. It is a good approximation for SPDC, however, because the signal and idler outputs in typical operation are much weaker than the pump beam.

then find that the electric-field complex envelopes must obey

$$\begin{aligned}
&\frac{\partial^2}{\partial z^2}\Big(A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&- \quad \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S\\
&+ \quad n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + n_P^2(\omega_P)A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&- \quad \frac{\chi^{(2)}}{c^2}\frac{\partial^2}{\partial t^2}\Big(A_I^*(z)A_P e^{-j[(\omega_P-\omega_I)t-(k_P-k_I)z]}\vec{i}_S\\
&+ \quad A_S^*(z)A_P e^{-j[(\omega_P-\omega_S)t-(k_P-k_S)z]}\vec{i}_I\Big) = \vec{0}. \qquad (24)
\end{aligned}$$

Performing the z differentiations on the first line of Eq. (24) gives

$$\begin{aligned}
&\frac{\partial^2}{\partial z^2}\Big(A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&= \quad \left[-k_S^2 A_S(z) + 2jk_S\frac{\partial A_S(z)}{\partial z}\right]e^{-j(\omega_S t-k_S z)}\vec{i}_S\\
&+ \quad \left[-k_I^2 A_I(z) + 2jk_I\frac{\partial A_I(z)}{\partial z}\right]e^{-j(\omega_I t-k_I z)}\vec{i}_I - k_P^2 A_P e^{-j(\omega_P-k_P z)}\vec{i}_P, \qquad (25)
\end{aligned}$$

where we have employed the slowly-varying envelope approximation to suppress terms involving $\frac{\partial^2}{\partial z^2}A_m(z)$ for $m = S, I$. Performing the t differentiations on the second and third lines of Eq. (24) yields

$$\begin{aligned}
&-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(n_S^2(\omega_S)A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + n_I^2(\omega_I)A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I\\
&+ \quad n_P^2(\omega_P)A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P\Big)\\
&= \quad k_S^2 A_S(z)e^{-j(\omega_S t-k_S z)}\vec{i}_S + k_I^2 A_I(z)e^{-j(\omega_I t-k_I z)}\vec{i}_I + k_P^2 A_P e^{-j(\omega_P t-k_P z)}\vec{i}_P, \qquad (26)
\end{aligned}$$

where we have used $k_m = \omega_m n_m(\omega_m)/c$ for $m = S, I, P$. Using Eqs. (25) and (26) in Eq. (24) leads to term cancellations[5] that reduce the latter equation to

$$\begin{aligned}
&\left(2jk_S\frac{\partial A_S(z)}{\partial z}e^{-j(\omega_S t-k_S z)} + \frac{\chi^{(2)}\omega_S^2}{c^2}A_I^*(z)A_P e^{-j[\omega_S t-(k_P-k_I)z]}\right)\vec{i}_S\\
&+ \quad \left(2jk_I\frac{\partial A_I(z)}{\partial z}e^{-j(\omega_I t-k_I z)} + \frac{\chi^{(2)}\omega_I^2}{c^2}A_S^*(z)A_P e^{-j[\omega_I t-(k_P-k_S)z]}\right)\vec{i}_I = \vec{0}, \qquad (27)
\end{aligned}$$

[5]These cancellations are to be expected, as the terms in question are those for a linear dielectric and $k_m = \omega_m n_m(\omega_m)/c$ gives the plane-wave solutions for such media.

where we have used $\omega_P = \omega_S + \omega_I$.

We will be interested in SPDC systems in which the signal and idler are in orthogonal linear polarizations. In this case, the preceding equation can be decomposed into two *coupled-mode* equations:[6]

$$\frac{\partial A_S(z)}{\partial z} = j\frac{\omega_S \chi^{(2)} A_P}{2cn_S(\omega_S)} A_I^*(z) e^{j\Delta kz} \tag{28}$$

$$\frac{\partial A_I(z)}{\partial z} = j\frac{\omega_I \chi^{(2)} A_P}{2cn_I(\omega_I)} A_S^*(z) e^{j\Delta kz}, \tag{29}$$

for $0 \leq z \leq l$, where $\Delta k \equiv k_P - k_S - k_I$. Equations (28) and (29) should be solved subject to given initial conditions at $z = 0$, i.e., given values for $A_S(0)$ and $A_I(0)$. Once $A_S(l)$ and $A_I(l)$ are found, the resulting electric field for $z > l$ is then

$$\begin{aligned}\vec{E}(z,t) &= \text{Re}[A_S(l)e^{-j(\omega_S t - k_S l - \omega_S(z-l)/c)}]\vec{i}_S + \text{Re}[A_I(l)e^{-j(\omega_I t - k_I l - \omega_I(z-l)/c)}]\vec{i}_I \\ &+ \text{Re}[A_P e^{-j(\omega_P t - k_P l - \omega_P(z-l)/c)}]\vec{i}_P,\end{aligned} \tag{30}$$

i.e., free-space plane-wave propagation prevails.[7] Here we can see why quantum mechanics is needed to properly understand the SPDC process shown on slide 3. If $A_S(0) = A_I(0) = 0$, in our classical analysis, then we get $A_S(l) = A_I(l) = 0$ from our coupled-mode equations,[8] and hence $\vec{E}(z,t) = \text{Re}[A_P e^{j(\omega_P t - k_P z)}]\vec{i}_P$ for $z > l$.

Solution to the Coupled-Mode Equations

So far we have been working with Maxwell's equations—and hence have developed coupled-mode equations—in SI units, i.e., the complex envelopes $A_S(z), A_I(z)$, and A_P have V/m units. Before solving the coupled-mode equations, it will be convenient for us to convert them to photon units, so as to ease the transition we will make—in Lecture 21—from the classical solution to the quantum version. The key to making this conversion is power flow.

Consider a monochromatic, $+z$-going plane wave in an isotropic linear dielectric whose electric and magnetic fields are

$$\vec{E}(z,t) = \text{Re}[Ae^{-j(\omega t - kz)}]\vec{i}_x \quad \text{and} \quad \vec{H}(z,t) = \text{Re}[c\epsilon_0 n(\omega) Ae^{-j(\omega t - kz)}]\vec{i}_y. \tag{31}$$

[6]If we regard the signal-frequency and idler-frequency components of the total field as *modes*, then these equations clearly couple them through the action of the strong pump beam and the crystal's $\chi^{(2)}$ nonlinearity.

[7]Our analysis assumes that anti-reflection coatings have been applied to the crystal's entrance and exit facets.

[8]If this statement is not immediately obvious, see the next section, in which we present solutions to the photon-units form of the coupled-mode equations

The time-average power (in W) crossing an area $\mathcal{A}$ in a constant-z plane is

$$S(z) = \frac{c\epsilon_0 n(\omega)\mathcal{A}}{2}\,|A|^2. \tag{32}$$

Were A to be written in $\sqrt{\text{photons/s}}$ units—for the chosen area $\mathcal{A}$—we would get[9]

$$S(z) = \hbar\omega|A|^2 \tag{33}$$

for the time-average power (in W) crossing the same area. It follows that

$$A|_{\sqrt{\text{photons/s}}} = \sqrt{\frac{c\epsilon_0\mathcal{A}}{2\hbar\omega}}\,A|_{\text{V/m}}. \tag{34}$$

Making this substitution in Eqs. (28) and (29) leads to the photon-units coupled-mode equations,

$$\frac{\partial A_S(z)}{\partial z} = j\kappa A_I^*(z)e^{j\Delta kz} \tag{35}$$

$$\frac{\partial A_I(z)}{\partial z} = j\kappa A_S^*(z)e^{j\Delta kz}, \tag{36}$$

for $0 \leq z \leq l$, where

$$\kappa \equiv \sqrt{\frac{\hbar\omega_S\omega_I\omega_P}{2c^3\epsilon_0 n_S(\omega_S)n_I(\omega_I)n_P(\omega_P)\mathcal{A}}}\,\chi^{(2)}A_P \tag{37}$$

is a complex-valued coupling constant that is proportional to the pump's complex envelope and the crystal's second-order nonlinear susceptibility.

The preceding photon-units coupled-mode equations have the following solution,

$$A_S(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_S(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_I^*(0)\right]e^{j\Delta kl/2} \tag{38}$$

$$A_I(l) = \left[\left(\cosh(pl) - \frac{j\Delta kl}{2}\frac{\sinh(pl)}{pl}\right)A_I(0) + j\kappa l\frac{\sinh(pl)}{pl}\,A_S^*(0)\right]e^{j\Delta kl/2}, \tag{39}$$

where

$$p \equiv \sqrt{|\kappa|^2 - (\Delta k/2)^2}, \tag{40}$$

as the reader may want to verify by substituting these results into the coupled-mode equations. Equations (38) and (39) have two interesting features that are worth

[9] We have chosen $\sqrt{\text{photons/s}}$ units, which require us to account for a cross-sectional area, to avoid needing an explicit area factor when we examine the continuous-time photodetection statistics of our SPDC model.

noting at this time. The first concerns phase matching. The second is a prelude to our quantum treatment of SPDC.

Inside the crystal, the monochromatic signal, idler, and pump beams—at frequencies ω_S, ω_I, and ω_P, respectively, propagate at their phase velocities, $v_m(\omega_m) = \omega_m/k_m$ for $m = S, I, P$. The nonlinear interaction governed by the coupled-mode equations Eqs. (35) and (36) is said to be *phase matched* when $\Delta k = 0$, i.e., when $\omega_P/v_P = \omega_S/v_S + \omega_I/v_I$. For a phase-matched system the coupled-mode equations simplify to

$$\frac{\partial A_S(z)}{\partial z} = j\kappa A_I^*(z) \quad \text{and} \quad \frac{\partial A_I(z)}{\partial z} = j\kappa A_S^*(z), \quad \text{for } 0 \le z \le l, \tag{41}$$

which shows that the phase angle of the coupling between the signal and idler remains the same throughout the interaction. On the other hand, when phase-matching is violated, the phase of the coupling between the signal and idler rotates as these fields propagate through the crystal. As a result, the solution to the phase-matched coupled-mode equations,

$$A_S(l) = \cosh(|\kappa|l)A_S(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_I^*(0) \tag{42}$$

$$A_I(l) = \cosh(|\kappa|l)A_I(0) + j\frac{\kappa}{|\kappa|}\sinh(|\kappa|l)A_S^*(0), \tag{43}$$

shows increasing amounts of signal-idler coupling with increasing $|\kappa|l$, i.e., with increasing pump power or crystal length. In contrast, far from phase matching—when $|\Delta k/2| \gg |\kappa|$—we get $p \approx j|\Delta k|/2$, whence

$$A_S(l) \approx \left[[\cos(\Delta kl/2) - j\sin(\Delta kl/2)]A_S(0) + j\kappa l\frac{\sin(\Delta kl/2)}{\Delta kl/2}\, A_I^*(0)\right]e^{j\Delta kl/2} \tag{44}$$

$$A_I(l) \approx \left[[\cos(\Delta kl/2) - j\sin(\Delta kl/2)]A_I(0) + j\kappa l\frac{\sin(\Delta kl/2)}{\Delta kl/2}\, A_S^*(0)\right]e^{j\Delta kl/2}, \tag{45}$$

which further reduce to

$$A_S(l) \approx A_S(0) \quad \text{and} \quad A_I(l) \approx A_I(0), \tag{46}$$

when $|\Delta kl/2| \gg 1$, i.e., when the crystal is long enough that the phase mismatch, $\Delta k \neq 0$, rotates the signal-idler coupling phase through many 2π cycles. Phase matching is critical to SPDC; in terms of photon fission, for every 10^6 pump photons, we may get only one signal-idler pair from a phase-matched interaction.

Photon fission is a good place to start our comments about the quantum form of the coupled-mode equations. We have already noted that $\omega_P = \omega_S + \omega_I$ is consistent with the photon-fission energy conservation principle: $\hbar\omega_P = \hbar\omega_S + \hbar\omega_I$. The momentum of a $+z$-going single photon at frequency ω is $+z$-directed with magnitude $\hbar\omega$. Thus our phase-matching condition, $k_P = k_S + k_I$, becomes photon-fission momentum conservation, $\hbar k_P = \hbar k_S + \hbar k_I$, when applied at the single-photon level. Photons being produced in pairs smacks of the two-mode parametric amplifier that we studied earlier in the semester. That system was governed by a two-mode Bogoliubov transformation,

$$\hat{a}_S^{\text{out}} = \mu \hat{a}_S^{\text{in}} + \nu \hat{a}_I^{\text{in}\dagger} \quad \text{and} \quad \hat{a}_I^{\text{out}} = \mu \hat{a}_I^{\text{in}} + \nu \hat{a}_S^{\text{in}\dagger}, \quad \text{where } |\mu|^2 - |\nu|^2 = 1. \tag{47}$$

Comparing Eq. (47) with Eqs. (42) and (43) reveals a great similarity. Indeed, if we change field complex envelopes and their conjugates to annihilation operators and creation operators, respectively, the latter two equations become a two-mode Bogoliubov transformation with[10]

$$\mu \equiv \cosh(|\kappa| l) \quad \text{and} \quad \nu \equiv j \frac{\kappa}{|\kappa|} \sinh(|\kappa| l). \tag{48}$$

The Road Ahead

In the next lecture we shall develop the quantum treatment of SPDC and the optical parametric amplifier (OPA), which is SPDC enhanced by placing the nonlinear crystal inside a resonant optical cavity. We shall also begin studying the non-classical behavior that can be seen in continuous-time photodetection using the outputs from SPDC and the OPA.

[10]Even for the general case of $\Delta k \neq 0$, changing the field complex envelopes and their conjugates into annihilation and creation operators, respectively, converts the classical coupled-mode input-output relation into a two-mode Bogoliubov transformation. When $|\Delta k l/2| \gg 1$, however, that two-mode Bogoliubov transformation will have $\mu \approx 1$ and $\nu \approx 0$.