6.453 Quantum Optical Communication
Spring 2009

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Lecture Number 13
Fall 2008

Jeffrey H. Shapiro

Date: Thursday, October 23, 2008

The two-mode parametric amplifier and photon twins. The dual parametric amplifier and polarization entanglement.

Introduction

In this lecture will continue our study of parametric amplification. We will pick up where we left off last time, by examining the entangled state that is produced by a two-mode parametric amplifier when its input modes are in their vacuum states. Despite the joint state of the output modes being non-classical—because entanglement is a non-classical property—we'll see that the individual modes are in classical states. We will develop the number-ket representations for both the joint and the individual states, and show that the former exhibits photon-twinning behavior, which leads to a non-classical signature in differenced direct detection. By appropriately ganging together a pair of two-mode parametric amplifiers, and going to the low-gain limit, we will show how to create a pair of polarization-entangled photons. Polarization entanglement provides the basis for qubit teleportation, which we will mention briefly at the end of today's lecture and treat in detail next time.

The Two-Mode Parametric Amplifier

From Lecture 12 we have the following two-mode Bogoliubov transformation relating the annihilation operators for the x- and y-polarized input and output modes, where we have chosen to specialize the general case to $\sqrt{G} = \mu > 0$ and $\sqrt{G-1} = \nu > 0$:

$$\hat{a}_{\text{out}_x} = \sqrt{G}\,\hat{a}_{\text{in}_x} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_y} \quad \text{and} \quad \hat{a}_{\text{out}_y} = \sqrt{G}\,\hat{a}_{\text{in}_y} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_x}. \tag{1}$$

We know that this transformation preserves annihilation operator commutator brackets, and that if we regard $\hat{a}_{\text{in}_x}$ and $\hat{a}_{\text{out}_x}$ as input and output with $\hat{a}_{\text{in}_y}$ in its vacuum state, the first equality in (1) implies that we have a phase-insensitive linear amplifier relation between the input and output modes. The same behavior occurs—i.e., we have a phase-insensitive linear amplifier relationship—if we take $\hat{a}_{\text{in}_y}$ and $\hat{a}_{\text{out}_y}$ as input and output with $\hat{a}_{\text{in}_x}$ in its vacuum state. In the $\pm 45°$ basis, however, (1) reduces to a pair of single-mode Bogoliubov transformations, so that a coherent-state input in the $\hat{a}_{\text{in}_+}$ ($\hat{a}_{\text{in}_-}$) mode will yield a squeezed-state output in the $\hat{a}_{\text{out}_+}$ ($\hat{a}_{\text{out}_-}$) mode. Thus, in the $\pm 45°$ basis, the two-mode parametric amplifier specified by (1) is a pair of independent phase-sensitive amplifiers, one for the $+45°$-polarized mode and the other for the $-45°$-polarized mode.

In Lecture 12, we used characteristic functions to derive the complete statistics for the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes and found the following result for the anti-normally ordered characteristic function associated with their joint density operator $\hat{\rho}_{\text{out}}$:

$$\chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = \chi_W^{\rho_{\text{in}}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y) e^{-(|\zeta_x|^2 + |\zeta_y|^2)/2}, \tag{2}$$

where

$$\xi_x = \sqrt{G}\,\zeta_x - \sqrt{G-1}\,\zeta_y^* \quad \text{and} \quad \xi_y = \sqrt{G}\,\zeta_y - \sqrt{G-1}\,\zeta_x^*. \tag{3}$$

Our interest lies in the special case in which the two input modes are in their vacuum states, so that the preceding anti-normally ordered characteristic function reduces to

$$\chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = e^{-G(|\zeta_x|^2 + |\zeta_y|^2) + 2\sqrt{G(G-1)}\,\text{Re}(\zeta_x \zeta_y)}, \tag{4}$$

from which the following anti-normally ordered characteristic functions for $\hat{\rho}_{\text{out}_x}$ and $\hat{\rho}_{\text{out}_y}$ readily follow:

$$\chi_A^{\rho_{\text{out}_x}}(\zeta_x^*, \zeta_x) = \chi_A^{\rho_{\text{out}}}(\zeta_x^*, 0, \zeta_x, 0) = e^{-G|\zeta_x|^2}, \tag{5}$$

and

$$\chi_A^{\rho_{\text{out}_y}}(\zeta_y^*, \zeta_y) = \chi_A^{\rho_{\text{out}}}(0, \zeta_y^*, 0, \zeta_y) = e^{-G|\zeta_y|^2}. \tag{6}$$

Thus, because

$$\chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) \neq \chi_A^{\rho_{\text{out}_x}}(\zeta_x^*, \zeta_x)\chi_A^{\rho_{\text{out}_y}}(\zeta_y^*, \zeta_y), \tag{7}$$

we have $\hat{\rho}_{\text{out}} \neq \hat{\rho}_{\text{out}_x} \otimes \hat{\rho}_{\text{out}_y}$, which means that $\hat{\rho}_{\text{out}}$ is an entangled state *if* it is a pure state. To show that $\hat{\rho}_{\text{out}}$ is a pure state we could argue that (1) is a unitary transformation, so that its output state must be a pure state when its input state—in this case the vacuum state of both the $\hat{a}_{\text{in}_x}$ and $\hat{a}_{\text{in}_y}$ modes—is a pure state. We shall take a more explicit route to showing that $\hat{\rho}_{\text{out}}$ implied by (4) is a pure state, viz., we shall verify that $\text{tr}(\hat{\rho}_{\text{out}}^2) = 1$.

Using the operator-valued inverse Fourier transform relation

$$\hat{\rho}_{\text{out}} = \int \frac{d^2\zeta_x}{\pi} \int \frac{d^2\zeta_y}{\pi}\, \chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) e^{-\zeta_x \hat{a}_{\text{out}_x}^\dagger - \zeta_y \hat{a}_{\text{out}_y}^\dagger} e^{\zeta_x^* \hat{a}_{\text{out}_x} + \zeta_y^* \hat{a}_{\text{out}_y}}, \tag{8}$$

we find that

$$\operatorname{tr}(\hat{\rho}_{\text{out}}^2) = \int \frac{d^2\alpha_x}{\pi} \int \frac{d^2\alpha_y}{\pi} \, {}_y\langle \alpha_y|_x\langle \alpha_x|\hat{\rho}_{\text{out}}^2|\alpha_x\rangle_x|\alpha_y\rangle_y \tag{9}$$

$$= \int \frac{d^2\zeta_x}{\pi} \int \frac{d^2\zeta_y}{\pi} \, \chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) \operatorname{tr}(\hat{\rho}_{\text{out}} e^{-\zeta_x \hat{a}_{\text{out}_x}^\dagger - \zeta_y \hat{a}_{\text{out}_y}^\dagger} e^{\zeta_x^* \hat{a}_{\text{out}_x} + \zeta_y^* \hat{a}_{\text{out}_y}}) \tag{10}$$

$$= \int \frac{d^2\zeta_x}{\pi} \int \frac{d^2\zeta_y}{\pi} \, \chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) \chi_N^{\rho_{\text{out}}}(-\zeta_x^*, -\zeta_y^*, -\zeta_x, -\zeta_y) \tag{11}$$

$$= \int \frac{d^2\zeta_x}{\pi} \int \frac{d^2\zeta_y}{\pi} \, \chi_W^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) \chi_W^{\rho_{\text{out}}}(-\zeta_x^*, -\zeta_y^*, -\zeta_x, -\zeta_y) \tag{12}$$

$$= \int \frac{d^2\zeta_x}{\pi} \int \frac{d^2\zeta_y}{\pi} \, |\chi_W^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y)|^2. \tag{13}$$

Here: the trace is evaluated in the first equality using the coherent-state bases $\{|\alpha_x\rangle_x\}$ and $\{|\alpha_y\rangle_y\}$; $\chi_N^{\rho_{\text{out}}}$ and $\chi_W^{\rho_{\text{out}}}$ are the normally-ordered and Wigner characteristic functions of the output state; the fourth equality makes use of the Baker-Campbell-Hausdorff theorem; and the last equality follows from $\chi_W^{\rho_{\text{out}}}$ being an Hermitian function of its arguments. Substituting

$$\chi_W^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = \chi_A^{\rho_{\text{out}}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) e^{(|\zeta_x|^2 + |\zeta_y|^2)/2} \tag{14}$$

$$= e^{-(G-1/2)(|\zeta_x|^2 + |\zeta_y|^2) + 2\sqrt{G(G-1)}\,\operatorname{Re}(\zeta_x \zeta_y)}, \tag{15}$$

into (13) we get

$$\operatorname{tr}(\hat{\rho}_{\text{out}}^2) = \int \frac{d^2\zeta_x}{\pi} \int \frac{d^2\zeta_y}{\pi} \, e^{-(2G-1)(|\zeta_x|^2 + |\zeta_y|^2) + 4\sqrt{G(G-1)}\,\operatorname{Re}(\zeta_x \zeta_y)} = 1, \tag{16}$$

where the second equality follows from the normalization integral for a 4-D Gaussian probability density function,[1] proving that the output state of the two-mode parametric amplifier is a pure state when its input modes are in their vacuum states. So, because there must be a $|\psi\rangle_{\text{out}}$ on the joint state space of the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes such that $\hat{\rho}_{\text{out}} = |\psi\rangle_{\text{out}\,\text{out}}\langle\psi|$, (7) implies that there are no pure states $|\psi\rangle_{\text{out}_x}$ and $|\psi\rangle_{\text{out}_y}$ for the individual modes which give $|\psi\rangle_{\text{out}} = |\psi\rangle_{\text{out}_x} \otimes |\psi\rangle_{\text{out}_y}$. In short, the output modes from the two-mode parametric amplifier are *entangled* when their input modes are in their vacuum states because if their joint state is a product state—i.e., unentangled—and pure, then the individual states must also be pure.

Let us delve deeper into the individual and joint states whose characteristic functions we've just determined. First consider the individual states. Our work in Lecture 12 on the phase-insensitive amplifier immediately tells us that the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$

[1]See (24), below, for the necessary formula.

modes are in classical states, with the following P-representations,

$$\hat{\rho}_{\text{out}_k} = \int \mathrm{d}^2\alpha \, \frac{e^{-|\alpha|^2/(G-1)}}{\pi(G-1)} \, |\alpha\rangle\langle\alpha|, \quad \text{for } k = x, y, \tag{17}$$

i.e., they are classically-random mixtures of coherent states with pdfs

$$P_{\text{out}_x}(\alpha, \alpha^*) = P_{\text{out}_y}(\alpha, \alpha^*) = \frac{e^{-|\alpha|^2/(G-1)}}{\pi(G-1)}. \tag{18}$$

Even though the individual x- and y-polarized modes are in mixed states, their joint density operator is a pure state. That the individual states are mixed, when their joint state is pure, is a signature of entanglement.

To get another perspective on the last remark, let's represent $\hat{E}_{\text{out}}(t)$ in the $\pm 45°$ basis, instead of the x-y basis. In this case we know that $\hat{a}_{\text{out}_\pm}$ is related to $\hat{a}_{\text{in}_\pm}$ by a single-mode Bogoliubov transformation. Thus each mode transforms its vacuum state input into a squeezed-vacuum state output. In particular, because the $\pm 45°$ inputs are in their vacuum states, and because

$$\hat{a}_{\text{in}_+} = \sqrt{G}\,\hat{a}_{\text{out}_+} - \sqrt{G-1}\,\hat{a}^\dagger_{\text{out}_+} \quad \text{and} \quad \hat{a}_{\text{in}_-} = \sqrt{G}\,\hat{a}_{\text{out}_-} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{out}_-}, \tag{19}$$

our prior work on squeezed states tells us that the joint output state in the diagonal ($\pm 45°$) basis is the following tensor product of squeezed vacuum states:

$$|\psi\rangle_{\text{out}} = |\psi\rangle_{\text{out}_+}|\psi\rangle_{\text{out}_-} = |0; \sqrt{G}, -\sqrt{G-1}\rangle_{\text{out}_+}|0; \sqrt{G}, \sqrt{G-1}\rangle_{\text{out}_-}. \tag{20}$$

Here the joint state is pure—as it must be *regardless* of which basis we use to represent it, because (1) implies a unitary state transformation and we are starting from a joint state that is pure—*and* the individual states are also pure, because the $\hat{a}_{\text{out}_\pm}$ modes are *not* entangled.

Our next task will be to focus on the number-ket representations of the joint and individual states for the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes, as these representations will be essential to our understanding of photon twinning behavior and polarization entanglement. Before doing so, however, a word about going from joint statistics to individual statistics is in order. Let $\hat{\rho}_{ab}$ be the density operator (joint state) for two electromagnetic modes whose annihilation operators are $\hat{a}$ and $\hat{b}$, respectively, and let $\hat{\rho}_a$ and $\hat{\rho}_b$ be their individual states. Last time we saw how to obtain the anti-normally ordered characteristic functions for the individual states given the anti-normally ordered characteristic function for their joint density operator. All we did there was to recognize that

$$\chi_A^{\rho_a}(\zeta_a^*, \zeta_a) = \chi_A^{\rho_{ab}}(\zeta_a^*, \zeta_b^*, \zeta_a, \zeta_b)|_{\zeta_b=0} \text{ and } \chi_A^{\rho_b}(\zeta_b^*, \zeta_b) = \chi_A^{\rho_{ab}}(\zeta_a^*, \zeta_b^*, \zeta_a, \zeta_b)|_{\zeta_a=0}, \tag{21}$$

follows immediately from the definitions of these characteristic functions. In the density operator domain, equivalent results are obtained by tracing out the unwanted modes, i.e.,

$$\hat{\rho}_a = \mathrm{tr}_b(\hat{\rho}_{ab}) = \sum_n {}_b\langle\phi_n|\hat{\rho}_{ab}|\phi_n\rangle_b \quad \text{and} \quad \hat{\rho}_b = \mathrm{tr}_a(\hat{\rho}_{ab}) = \sum_n {}_a\langle\phi_n|\hat{\rho}_{ab}|\phi_n\rangle_a, \tag{22}$$

where $\{|\phi_n\rangle_a\}$ and $\{|\phi_n\rangle_b\}$ are arbitrary complete-orthonormal sets of kets on $\mathcal{H}_a$ and $\mathcal{H}_b$, the state spaces of the $\hat{a}$ and $\hat{b}$ modes, respectively.

It is conventional to refer to $\hat{\rho}_a$ and $\hat{\rho}_b$ as *reduced* density operators. Note that—because $\hat{\rho}_{ab}$ is defined on the tensor product state space $\mathcal{H}_a\otimes\mathcal{H}_b$—we have ${}_a\langle\phi_n|\hat{\rho}_{ab}|\phi_n\rangle_a$ is an operator on $\mathcal{H}_b$ and ${}_b\langle\phi_n|\hat{\rho}_{ab}|\phi_n\rangle_b$ is an operator on $\mathcal{H}_a$. Finally, remember that the trace operation does *not* need to be performed using an orthonormal basis. An overcomplete basis that resolves the identity operator will also do, e.g., we have that

$$\hat{\rho}_b = \int \frac{\mathrm{d}^2\alpha}{\pi}\, {}_a\langle\alpha|\hat{\rho}_{ab}|\alpha\rangle_a \quad \text{and} \quad \hat{\rho}_a = \int \frac{\mathrm{d}^2\beta}{\pi}\, {}_b\langle\beta|\hat{\rho}_{ab}|\beta\rangle_b, \tag{23}$$

where $\{|\alpha\rangle_a\}$ and $\{|\beta\rangle_b\}$ are the coherent states of the $\hat{a}$ and $\hat{b}$ modes, respectively, cf. what we did in (9).

Number-Ket Representation for the Output State of the Two-Mode Parametric Amplifier

To better understand—and, more importantly, to see how to usefully employ—the entangled state of the $\hat{a}_{\mathrm{out}_x}$ and $\hat{a}_{\mathrm{out}_y}$ modes that results when our two-mode parametric amplifier has its $\hat{a}_{\mathrm{in}_x}$ and $\hat{a}_{\mathrm{in}_y}$ modes in their vacuum states, we need to develop the number-ket representation for the output state. Toward this end, we first give the normally-ordered form of the density operator, $\hat{\rho}_{\mathrm{out}}$. From classical probability theory, we have the following inverse transform relation linking the joint characteristic function for an N-D zero-mean, real-valued Gaussian random vector, $\mathbf{x}$, to its joint probability probability density,

$$\frac{\exp(-\mathbf{x}^T\Lambda^{-1}\mathbf{x}/2)}{(2\pi)^{N/2}|\Lambda|^{1/2}} = \int \frac{\mathrm{d}\mathbf{v}}{(2\pi)^N}\, \exp(-\mathbf{v}^T\Lambda\mathbf{v}/2). \tag{24}$$

Here, $\mathbf{x}$ and $\mathbf{v}$ are N-D column vectors, and Λ is the covariance matrix of $\mathbf{x}$, with $|\Lambda|$ being its determinant. Applying this formula to (4) yields—after some tedious algebra that will be omitted—the following expression for the normally-ordered form of the density operator,

$$\rho^{(n)}_{\mathrm{out}}(\alpha_x^*, \alpha_y^*, \alpha_x., \alpha_y) \equiv {}_x\langle\alpha_x|_y\langle\alpha_y|\hat{\rho}_{\mathrm{out}}|\alpha_x\rangle_x|\alpha_y\rangle_y \tag{25}$$

$$= \frac{e^{-|\alpha_x|^2-|\alpha_y|^2+2\sqrt{(G-1)/G}\,\mathrm{Re}(\alpha_x\alpha_y)}}{G}. \tag{26}$$

We already know that $\hat{\rho}_{\text{out}}$ is a pure state, $|\psi\rangle_{\text{out}}$. We now claim that this state has the following number-ket representation,

$$|\psi\rangle_{\text{out}} = \sum_{n=0}^{\infty} \sqrt{\frac{(G-1)^n}{G^{n+1}}}\, |n\rangle_x |n\rangle_y. \tag{27}$$

Because $\hat{\rho}_{\text{out}}$ is determined by its normally-ordered form, to verify that $\hat{\rho}_{\text{out}} = |\psi\rangle_{\text{out}\,\text{out}}\langle\psi|$ with $|\psi\rangle_{\text{out}}$ given by (27), it suffices to verify that

$$|_x\langle\alpha_x|_y\langle\alpha_y|\psi\rangle_{\text{out}}|^2 = \frac{e^{-|\alpha_x|^2-|\alpha_y|^2+2\sqrt{(G-1)/G}\,\text{Re}(\alpha_x\alpha_y)}}{G}. \tag{28}$$

But, this verification is simple, because

$$\begin{aligned} {}_x\langle\alpha_x|_y\langle\alpha_y|\psi\rangle_{\text{out}} &= \sum_{n=0}^{\infty} \sqrt{\frac{(G-1)^n}{G^{n+1}}}\, \frac{(\alpha_x^*\alpha_y^*)^n}{n!}\, e^{-|\alpha_x|^2/2-|\alpha_y|^2/2} && (29)\\ &= \frac{e^{-|\alpha_x|^2/2-|\alpha_y|^2/2+\sqrt{(G-1)/G}\,\alpha_x^*\alpha_y^*}}{\sqrt{G}}. && (30) \end{aligned}$$

Taking the magnitude squared of this last expression then completes the desired verification. We shall explore the physics of this result in the next section. For now we just note that $|\psi\rangle_{\text{out}}$ is explicitly an entangled state, because its number-ket representation cannot be factored into the form $|\psi\rangle_{\text{out}_x} \otimes |\psi\rangle_{\text{out}_y}$ with

$$|\psi\rangle_{\text{out}_k} = \sum_{n=0}^{\infty} \psi_{k_n} |n\rangle_k, \quad \text{for } k = x, y, \tag{31}$$

as we have assumed $G > 1$.

Photon-Twinning Behavior

Equation (27) shows that the x- and y-polarized outputs from a two-mode parametric amplifier, whose input modes are in their vacuum states, are in a pure entangled state that is the superposition of states in which the x- and y-polarized output modes each have the *same* number of photons. When we study second-order nonlinear optics later this term, we will get a good physical picture for why these outputs should each have the same number of photons. For now, let's just see how this "photon twinning" behavior manifests itself in direct detection measurements.

Consider the photodetection setup shown in Slide 7. Here, the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes illuminate a polarizing beam splitter, which has the effect of directing the

$\hat{a}_{\text{out}_x}$ mode to one detector and the $\hat{a}_{\text{out}_y}$ mode to another detector.[2] The photocount difference for the two detectors is our measurement quantity of interest, i.e., we are concerned with the observable $\hat{N} \equiv \hat{N}_y - \hat{N}_x$, where

$$\hat{N}_x \equiv \hat{a}^\dagger_{\text{out}_x}\hat{a}_{\text{out}_x} \quad \text{and} \quad \hat{N}_y \equiv \hat{a}^\dagger_{\text{out}_y}\hat{a}_{\text{out}_y}. \tag{32}$$

It is easy to see that, for $n = 0, 1, 2, \ldots$, the photon-twin number state $|n\rangle_x|n\rangle_y$ is an eigenket of $\hat{N}$ with eigenvalue zero. In simple terms, this just says that if each of these modes has exactly n photons and we count photons in each mode, then the difference between the two photocount measurements is *always* zero. Now, because $|\psi\rangle_{\text{out}}$ is a superposition of $|n\rangle_x|n\rangle_y$, it too is a zero-eigenvalue eigenket of $\hat{N}$. Thus quantum photodetection theory predicts that

$$\langle \Delta\hat{N}^2\rangle = 0, \tag{33}$$

for the measurement setup shown in Slide 7.

What does semiclassical theory say about the variance of the photocount difference? Suppose that the first detector is illuminated by a single-mode classical field with complex phasor a_{out_x} and the other photodetector is illuminated by a single-mode classical field with complex phasor a_{out_y}. Because excess noise can never decrease photocount fluctuations below shot-noise levels, we shall assume that a_{out_x} and a_{out_y} are both deterministic, so that N_x and N_y have the shot-noise limited variances

$$\langle \Delta N_x^2\rangle = \langle N_x\rangle = |a_{\text{out}_x}|^2 \quad \text{and} \quad \langle \Delta N_y^2\rangle = \langle N_y\rangle = |a_{\text{out}_y}|^2. \tag{34}$$

Because the shot noises on physically separate detectors are statistically independent we have that

$$\langle \Delta N^2\rangle = \langle \Delta N_x^2\rangle + \langle \Delta N_y^2\rangle = \langle N_x\rangle + \langle N_y\rangle \geq 0, \tag{35}$$

where the inequality is strict unless $a_{\text{out}_x} = a_{\text{out}_y} = 0$, i.e., unless the measurement setup is not illuminated. Comparison of Eqs. (33) and (35) shows that we have identified a non-classical signature, i.e., the photon twins behavior exhibited by the state $|\psi\rangle_{\text{out}}$ from our two-mode parametric amplifier *cannot* be explained by semiclassical photodetection theory.

We know from Lecture 12 that $\hat{\rho}_{\text{out}_x}$ and $\hat{\rho}_{\text{out}_y}$ both have proper P-representations—i.e., they are individually in classical states—so that semiclassical photodetection theory suffices to describe all measurements made on the $\hat{a}_{\text{out}_x}$ or $\hat{a}_{\text{out}_y}$ modes alone. The photon twins behavior we have just demonstrated shows that the entanglement between these two modes precludes the use of semiclassical photodetection theory to

[2] The polarizing beam splitter also couples a y-polarized vacuum-state mode to the first detector and an x-polarized vacuum-state mode to the second detector. These modes enter through the beam splitter's unused input port. However, because they do not contribute to the photocounts obtained from these two detectors, we have omitted them from Slide 7, and will not carry them along in our analysis.

quantify joint measurements, i.e., measurements that sense both of these modes. Before moving on to the generation of polarization entanglement from a pair of two-mode parametric amplifiers, let us conclude this section by deriving the number-ket representations of the reduced density operators $\hat{\rho}_{\text{out}_x}$ and $\hat{\rho}_{\text{out}_y}$. These are easily found, by using $\hat{\rho}_{\text{out}} = |\psi\rangle_{\text{out}\,\text{out}}\langle\psi|$ in conjunction with (27) and tracing out the unwanted modes using the number-ket basis. We find that

$$\hat{\rho}_{\text{out}_x} = \sum_{n=0}^{\infty} \frac{(G-1)^n}{G^{n+1}}\,|n\rangle_{x\,x}\langle n| \quad \text{and} \quad \hat{\rho}_{\text{out}_y} = \sum_{n=0}^{\infty} \frac{(G-1)^n}{G^{n+1}}\,|n\rangle_{y\,y}\langle n|. \tag{36}$$

These results say that the individual output-mode states are Bose-Einstein mixtures of photon-number states.[3] But, from (17) we see that the individual output-mode states are *also* zero-mean, complex-valued Gaussian mixtures of coherent states.

You should not be confused by there being two equivalent probabilistic interpretations each for $\hat{\rho}_{\text{out}_x}$ and $\hat{\rho}_{\text{out}_y}$. Consider a mixed state $\hat{\rho} = \sum_n p_n|\phi_n\rangle\langle\phi_n|$, where $\{p_n\}$ is a probability distribution. This form of the density operator has the following interpretation: the state of the system is $|\phi_n\rangle$ with probability p_n. There is no constraint on collection of states $\{|\phi_n\rangle\}$, viz., they need *not* be orthonormal. From the homework we know that $\hat{\rho}$ has an eigenket-eigenvalue expansion $\hat{\rho} = \sum_n \rho_n|\rho_n\rangle\langle\rho_n|$ where the $\{|\rho_n\rangle\}$ are orthonormal and $\{\rho_n\}$ is a probability distribution. This alternate form of the density operator has the following interpretation: the state of the system is $|\rho_n\rangle$ with probability ρ_n. Both of these interpretations provide prescriptions for constructing a system in the given $\hat{\rho}$. No measurement made on the system—given it is in state $\hat{\rho}$—can distinguish between these two interpretations. As a check on the equivalence between our two representations for $\hat{\rho}_{\text{out}_x}$ and $\hat{\rho}_{\text{out}_y}$, let us use their number-ket representations to verify their normally-ordered forms. We have that

$$\rho^{(n)}(\alpha_k^*, \alpha_k) = {}_k\langle\alpha_k|\hat{\rho}_{\text{out}_k}|\alpha_k\rangle_k = \sum_{n=0}^{\infty} \frac{(G-1)^n}{G^{n+1}}\,|{}_k\langle\alpha_k|n\rangle_k|^2 \tag{37}$$

$$= \sum_{n-0}^{\infty} \frac{(G-1)^n}{G^{n+1}}\,\frac{|\alpha_k|^{2n}}{n!}\,e^{-|\alpha_k|^2} = \frac{e^{-|\alpha_k|^2/G}}{G}, \quad \text{for } k = x, y, \tag{38}$$

as expected.

Generating Polarization Entangled Photons

Consider the system shown in Slide 9. Here we have a pair of two-mode parametric amplifiers, of the type we have been studying. Their output field operators are

$$\hat{\mathbf{E}}_{\text{out}_K}(t) = \frac{\hat{a}_{\text{out}_{K_x}} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_x + \frac{\hat{a}_{\text{out}_{K_y}} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_y, \qquad \text{for } K = A, B \text{ and } 0 \le t \le T, \tag{39}$$

[3] Recall that the Bose-Einstein probability mass function with mean $\bar{N}$ is $\Pr(N = n) = \bar{N}^n/(\bar{N} + 1)^{n+1}$, for $n = 0, 1, 2, \ldots$

where we have suppressed the vacuum-state "other modes." We will assume that the input modes to these parametric amplifiers are all in their vacuum states, and that their pumps[4] are phased such that

$$\hat{a}_{\text{out}_{A_x}} = \sqrt{G}\,\hat{a}_{\text{in}_{A_x}} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_{A_y}} \quad \text{and} \quad \hat{a}_{\text{out}_{A_y}} = \sqrt{G}\,\hat{a}_{\text{in}_{A_y}} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_{A_x}}, \tag{40}$$

for parametric amplifier A, while

$$\hat{a}_{\text{out}_{B_x}} = \sqrt{G}\,\hat{a}_{\text{in}_{B_x}} - \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_{B_y}} \quad \text{and} \quad \hat{a}_{\text{out}_{B_y}} = \sqrt{G}\,\hat{a}_{\text{in}_{B_y}} - \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_{B_x}}, \tag{41}$$

for parametric amplifier B, where $G > 1$.[5] We know that each parametric amplifier produces a pure state—of the type that we have studied earlier in this lecture—so that their joint output state is the tensor product of their individual states,

$$\begin{aligned} |\psi\rangle_{\text{out}} &= |\psi\rangle_A \otimes |\psi\rangle_B \\ &= \sum_{n=0}^{\infty} \sqrt{\frac{\Delta G^n}{G^{n+1}}}\,|n\rangle_{A_x}|n\rangle_{B_y} \otimes \sum_{m=0}^{\infty}(-1)^m \sqrt{\frac{\Delta G^m}{G^{m+1}}}\,|m\rangle_{B_x}|m\rangle_{A_y}, \end{aligned} \tag{42}$$

where $\Delta G \equiv G - 1$. The $(-1)^m$ factor in $|\psi\rangle_B$ can be verified by reprising the verification procedure for the number-ket representation of the parametric amplifier's output state, this time using the Bogoliubov transformation given above for amplifier B.

To generate polarization entanglement, the outputs from the parametric amplifiers A and B are combined on a polarizing beam splitter, so that operators for the fields emerging from the beam splitter's output ports are

$$\hat{\mathbf{E}}_{\text{out}\,1}(t) = \frac{\hat{a}_{\text{out}_{A_x}} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_x + \frac{\hat{a}_{\text{out}_{B_y}} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_y \tag{43}$$

$$\hat{\mathbf{E}}_{\text{out}\,2}(t) = \frac{\hat{a}_{\text{out}_{B_x}} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_x + \frac{\hat{a}_{\text{out}_{A_y}} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_y, \tag{44}$$

for $0 \le t \le T$. Now, if $\Delta G \ll 1$, so that we can truncate the sums in (42) to first order in ΔG, we get

$$|\psi\rangle_{\text{out}} \approx (|0\rangle_{A_x}|0\rangle_{B_y}) \otimes (|0\rangle_{B_x}|0\rangle_{A_y}) \tag{45}$$

$$+ \quad \sqrt{\Delta G}\,[(|1\rangle_{A_x}|0\rangle_{B_y}) \otimes (|0\rangle_{B_x}|1\rangle_{A_y}) - (|0\rangle_{A_x}|1\rangle_{B_y}) \otimes (|1\rangle_{B_x}|0\rangle_{A_y})], \tag{46}$$

[4]Later this term we will study enough nonlinear optics to understand the role of the pump field—not shown in the work we have done so far on parametric amplifiers—in parametric processes.

[5]This is a proper two-mode Bogoliubov transformation, as can be seen by choosing $\mu = \sqrt{G}$ and $\nu = -\sqrt{G-1}$.

where we have segregated terms in a way that makes clear which states are associated with $\hat{\mathbf{E}}_{\text{out}\,1}(t)$ and $\hat{\mathbf{E}}_{\text{out}\,2}(t)$. If we make photon counting measurements on these two fields, and only include—in our post-measurement analysis—cases in which we count a photon from each field, then we have performed a *post-selection* operation that corresponds to reducing the state in (46) to

$$|\psi\rangle_{\text{out}} = \frac{(|1\rangle_{A_x}|0\rangle_{B_y}) \otimes (|0\rangle_{B_x}|1\rangle_{A_y}) - (|0\rangle_{A_x}|1\rangle_{B_y}) \otimes (|1\rangle_{B_x}|0\rangle_{A_y})}{\sqrt{2}}, \tag{47}$$

where we have normalized to unit length. This state is the *singlet state* of the two-mode fields $\hat{\mathbf{E}}_{\text{out}\,1}(t)$ and $\hat{\mathbf{E}}_{\text{out}\,2}(t)$. Using $|\psi^-\rangle_{12}$ to denote this state, and introducing the following short-hand notations for x- and y-polarized single-photon states of the field operators $\hat{\mathbf{E}}_{\text{out}\,1}(t)$ and $\hat{\mathbf{E}}_{\text{out}\,2}(t)$,

$$|x\rangle_1 = |1\rangle_{A_x}|0\rangle_{B_y} \quad |y\rangle_1 = |0\rangle_{A_x}|1\rangle_{B_y} \quad |x\rangle_2 = |1\rangle_{B_x}|0\rangle_{A_y} \quad |y\rangle_2 = |0\rangle_{B_y}|1\rangle_{A_y}, \tag{48}$$

we can say that post-selected, low-gain operation of our two parametric amplifiers produces the singlet state

$$|\psi^-\rangle_{12} = \frac{|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2}{\sqrt{2}}. \tag{49}$$

The properties of this entangled state are extremely important in quantum information science, as we shall learn. Note that it has exactly two photons, one each associated with the fields $\hat{\mathbf{E}}_{\text{out}\,1}(t)$ and $\hat{\mathbf{E}}_{\text{out}\,2}(t)$.

At this point we are equipped to revisit the discussion of polarization entanglement that we presented in Lecture 1. This time, however, we are prepared for a complete analytical treatment. Slide 10 shows our measurement setup for polarization analysis. The field operators $\hat{\mathbf{E}}_{\text{out}\,1}(t)$ and $\hat{\mathbf{E}}_{\text{out}\,2}(t)$ illuminate a pair of these systems such that the $\mathbf{i} = \alpha\mathbf{i}_x + \beta\mathbf{i}_y$ polarization is converted—by wave plates—to the x polarization, and its orthogonal complement, the $\mathbf{i}' = \beta^*\mathbf{i}_x - \alpha^*\mathbf{i}_y$ polarization, is converted—by these same wave plates—to the y polarization. The polarizing beam splitter photon-counting modules then yield four outcomes, $\{N_{k_{\mathbf{i}}}, N_{k_{\mathbf{i}'}} : k = 1, 2\}$. When $N_{k_{\mathbf{i}}} = 1$ it means that the $\hat{\mathbf{E}}_{\text{out}\,k}(t)$ photon emerged from the wave plates polarized in the x direction, etc. As discussed in Lecture 1, we are interested in the conditional probabilities $\Pr(\, N_{2_{\mathbf{i}'}} = 1 \mid N_{1_{\mathbf{i}}} = 1\,)$ and $\Pr(\, N_{2_{\mathbf{i}}} = 1 \mid N_{1_{\mathbf{i}'}} = 1\,)$. To find these quantities we will use $|\mathbf{i}\rangle_k$ and $|\mathbf{i}'\rangle_k$ to denote $\mathbf{i}$- and $\mathbf{i}'$-polarized single-photon states of $\hat{\mathbf{E}}_{\text{out}\,k}(t)$, for $k = 1, 2$. We then have that

$$\Pr(N_{1_{\mathbf{i}}} = 1, N_{2_{\mathbf{i}'}} = 1) \;\; = \;\; |{}_1\langle\mathbf{i}|{}_2\langle\mathbf{i}'|\psi^-\rangle_{12}|^2 \tag{50}$$

$$\Pr(N_{1_{\mathbf{i}'}} = 1, N_{2_{\mathbf{i}}} = 1) \;\; = \;\; |{}_1\langle\mathbf{i}'|{}_2\langle\mathbf{i}|\psi^-\rangle_{12}|^2. \tag{51}$$

Writing the preceding inner products in terms of the x- and y-polarized single-photon states then yields

$$\Pr(N_{1_{\mathbf{i}}} = 1, N_{2_{\mathbf{i}'}} = 1) = \frac{|-\alpha^*\alpha - \beta^*\beta|^2}{2} = \frac{1}{2}. \tag{52}$$

$$\Pr(N_{1_{\mathbf{i}'}} = 1, N_{2_{\mathbf{i}}} = 1) = \frac{|\beta\beta^* + \alpha\alpha^*|^2}{2} = \frac{1}{2}. \tag{53}$$

Because these two probabilities sum to one, it follows that

$$\Pr(N_{1_{\mathbf{i}}} = 1, N_{2_{\mathbf{i}}} = 1) = 0 \quad \text{and} \quad \Pr(N_{1_{\mathbf{i}'}} = 1, N_{2_{\mathbf{i}'}} = 1) = 0. \tag{54}$$

Using this joint distribution for the photocount measurements, we can then show that the individual—$\hat{\mathbf{E}}_{\text{out}\,1}(t)$ and $\hat{\mathbf{E}}_{\text{out}\,2}(t)$—photocounts are completely random, i.e.,

$$\Pr(N_{k_{\mathbf{i}}} = 1) = \Pr(N_{k_{\mathbf{i}'}} = 1) = 1/2, \quad \text{for } k = 1, 2. \tag{55}$$

and this holds for all polarization bases $\{\mathbf{i}, \mathbf{i}'\}$. From the joint and the marginal statistics we now have the desired result: for *all* bases $\{\mathbf{i}, \mathbf{i}'\}$,

$$\Pr(\, N_{2_{\mathbf{i}'}} = 1 \mid N_{1_{\mathbf{i}}} = 1\,) = \Pr(\, N_{2_{\mathbf{i}}} = 1 \mid N_{1_{\mathbf{i}'}} = 1\,) = 1. \tag{56}$$

You should reread the Lecture 1 notes, where it is shown that the highest conditional probabilities that we can obtain from a classical theory of particle-like photons whose polarizations are individually random but completely correlated is

$$\Pr(\, N_{2_{\mathbf{i}'}} = 1 \mid N_{1_{\mathbf{i}}} = 1\,) = \Pr(\, N_{2_{\mathbf{i}}} = 1 \mid N_{1_{\mathbf{i}'}} = 1\,) = 2/3, \tag{57}$$

unless some sort of "action at a distance" is invoked.

So far, the polarization entanglement embodied by the single state $|\psi^-\rangle_{12}$ might be regarded as a cute quantum-mechanical parlor trick. Slide 12 is a reminder that there is at least one vitally important application for polarization entanglement: qubit teleportation. The polarization state of a single photon can be regarded as a quantum bit (qubit), i.e., an arbitrary unit-length superposition of a pair of orthonormal basis states, e.g., the x- and y-polarization states. Qubits must be communicated over long distances to network quantum computers. However, qubits are inherently fragile; 90% of the photons coupled into low-loss optical fiber will be lost after 50 km of propagation. Yet, by sharing entanglement between the end stations—shown as Alice and Bob on Slide 12—an arbitrary and unknown message qubit (provided by Charlie) may be successfully teleported from Alice to Bob.

The Road Ahead

In the next lecture we shall present the details of the qubit teleportation system shown in Slide 12. We shall then introduce and analyze a second kind of teleportation, i.e., one based on the continuous variables associated with the quadrature components of the annihilation operator. Here too entanglement will play a key role, although in this case it will be quadrature entanglement.