6.453 Quantum Optical Communication
Spring 2009

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Lecture Number 7
Fall 2008

Jeffrey H. Shapiro

Date: Thursday, September 25, 2008

Reading: For quantum characteristic functions:

- C.C. Gerry and P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005) Sect. 3.8.
- W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) Sect. 3.4. (On reserve at Barker library.)

For positive operator-valued measurements:

- M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000) Sect. 2.2.6.

Introduction

Today we will continue to explore the quadrature measurement statistics of the quantum harmonic oscillator, and use that exercise to introduce the notion of quantum characteristic functions. These characteristic functions, like their counterparts in classical probability theory, are useful calculational tools, as we will see when later when we study the quantum noise behavior of linear systems that have loss or gain, i.e., attenuators and amplifiers. Today we will also provide the positive operator-valued measurement (POVM) description for "measuring" the annihilation operator, $\hat{a}$. POVMs are extremely important in quantum information science, in that they are more general than observables. Lest you think that they are mere mathematical generalizations, it is worth noting now that, for a single-mode optical field, the $\hat{a}$ POVM has a physical realization: optical heterodyne detection.

Quadrature-Measurement Statistics

Tables 1 and 2 summarize most of what we have learned so far about quantum harmonic oscillator's quadrature-measurement statistics. Slides 4 and 5 give qualitative pictures of $\hat{a}_1(t)$ measurements when the oscillator is in a number state or coherent

State	$\langle \hat{a}(t) \rangle$
$\lvert n\rangle$	0
$\lvert \alpha\rangle$	$\alpha e^{-j\omega t}$
$\lvert \beta;\mu,\nu\rangle$	$(\mu^*\beta - \nu\beta^*)e^{-j\omega t}$

Table 1: Mean value of $\hat{a}(t)$ for number states, coherent states, and squeezed states. The real and imaginary parts of these table entries are the quadrature-measurement mean values, $\langle \hat{a}_1(t)\rangle$ and $\langle \hat{a}_2(t)\rangle$, respectively.

State	$\langle \Delta\hat{a}_1^2(t)\rangle$	$\langle \Delta\hat{a}_2^2(t)\rangle$
$\lvert n\rangle$	$(2n+1)/4$	$(2n+1)/4$
$\lvert \alpha\rangle$	$1/4$	$1/4$
$\lvert \beta;\mu,\nu\rangle$	$\lvert\mu - \nu e^{-2j\omega t}\rvert^2/4$	$\lvert\mu + \nu e^{-2j\omega t}\rvert^2/4$

Table 2: Quadrature-measurement variances for number states, coherent states, and squeezed states.

state (Slide 4), or an amplitude-squeezed or phase-squeezed state (Slide 5). In addition to these results, we also know—from our wave function analysis of minimum uncertainty-product states—that the probability density functions for the quadrature measurements are Gaussian, when the oscillator is in a coherent state or a squeezed state. We have yet to determine what this probability density is when the oscillator is in a number state, nor have we given a very clear and explicit description for the phase space pictures shown on Slides 4 and 5. We will remedy both of these deficiencies by means of quantum characteristic functions.

Quantum Characteristic Functions

It is appropriate to begin our discussion of quantum characteristic functions by stepping back to review what we know about classical characteristic functions. Suppose that x is a real-valued, classical random variable whose probability density function is $p_x(X)$.[1] The characteristic function of x,

$$M_x(jv) \equiv \langle e^{jvx}\rangle = \int_{-\infty}^{\infty} \mathrm{d}X\, e^{jvX} p_x(X), \tag{1}$$

[1] Probability density functions (pdfs) are natural ways to specify the statistics of a continuous random variable. However, if we allow pdfs to contain impulses, then they can be used for discrete random variables as well. Hence, although we have chosen to use pdf notation here, our remarks apply to *all* real-valued, classical random variables.

is equivalent to the pdf $p_x(X)$ in that it provides a complete statistical characterization of the random variable. This can be seen from the inverse relation,

$$p_x(X) = \int_{-\infty}^{\infty} \frac{\mathrm{d}v}{2\pi}\, M_x(jv)e^{-jvX}. \tag{2}$$

Indeed, Eqs. (1) and (2) show that $p_x(X)$ and $M_x(jv)$ are a Fourier transform pair. On Problem Set 1 you exercised the key properties of the classical characteristic function, so no further review will be given here. What we shall do is use the characteristic equation approach to determine the quadrature-measurement statistics of the harmonic oscillator.

Suppose that we measure

$$\hat{a}_1(t) = \mathrm{Re}[\hat{a}(t)] = \mathrm{Re}(\hat{a}e^{-j\omega t}) = \hat{a}_1\cos(\omega t) + \hat{a}_2\sin(\omega t), \tag{3}$$

where $\mathrm{Re}(\hat{a}) = \hat{a}_1$ and $\mathrm{Im}(\hat{a}) = \hat{a}_2$, for some fixed particular value of time, t. Let $a_1(t)$ denote the classical random variable that results from this measurement. Then the classical characteristic function for $a_1(t)$, when the oscillator's state is $|\psi\rangle$, can be found as follows,

$$M_{a_1(t)}(jv) = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, e^{jv\alpha_1} p_{a_1(t)}(\alpha_1) = \langle\psi| \left(\int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, e^{jv\alpha_1} |\alpha_1\rangle_{t_1 t_1}\langle\alpha_1|\right)|\psi\rangle, \tag{4}$$

where $\{|\alpha_1\rangle_{t_1}\}$ are the eigenkets of $\hat{a}_1(t)$, i.e., they are the (delta-function) orthonormal solutions to

$$\hat{a}_1(t)|\alpha_1\rangle_{t_1} = \alpha_1|\alpha_1\rangle_{t_1}, \quad \text{for } -\infty < \alpha_1 < \infty. \tag{5}$$

Expanding $e^{jv\alpha_1}$ in its Taylor series, we have that

$$M_{a_1(t)}(jv) = \langle\psi| \left(\int_{-\infty}^{\infty} \mathrm{d}\alpha_1 \sum_{n=0}^{\infty} \frac{(jv)^n}{n!}\, \alpha_1^n |\alpha_1\rangle_{t_1 t_1}\langle\alpha_1|\right)|\psi\rangle. \tag{6}$$

Next, we interchange the order of integration and summation, use the fact that the $\{|\alpha_1\rangle_{t_1}\}$ diagonalize $\hat{a}_1^n(t)$ for $n = 0, 1, 2, \ldots$, and obtain

$$M_{a_1(t)}(jv) = \langle\psi| \left(\sum_{n=0}^{\infty} \frac{(jv)^n}{n!}\, \hat{a}_1^n(t)\right)|\psi\rangle = \langle e^{jv\hat{a}_1(t)}\rangle, \tag{7}$$

where the exponential of an operator (here $\hat{a}_1(t)$) is *defined* by the Taylor series expansion. This final answer seems almost obvious. We said that $a_1(t)$ at time t is the classical random variable that results from measurement of $\hat{a}_1(t)$ at time t. It *should* follow that

$$\langle e^{jva_1(t)}\rangle = \langle e^{jv\hat{a}_1(t)}\rangle. \tag{8}$$

Nevertheless, you should bear in mind that the averaging on the left-hand side of this equation is a classical probability average, see (4), whereas the averaging on the right-hand side of this equation is a quantum average, see (7). We'll see more of this equivalence between the statistics of classical random variables and quantum operator measurements when we treat the quantum theory of photodetection.

We now introduce the Wigner characteristic function for $\hat{a}$, which is defined by

$$\chi_W(\zeta^*, \zeta) \equiv \langle e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger} \rangle, \tag{9}$$

for ζ a complex number whose real and imaginary parts are ζ_1 and ζ_2, respectively. From the second equality in Eq. (3) and rewriting Eq. (9) as

$$\chi_W(\zeta^*, \zeta) = \langle e^{-2j\mathrm{Im}[\zeta^* \hat{a}]} \rangle, \tag{10}$$

we can show that

$$M_{a_1(t)}(jv) = \chi_W(\zeta^*, \zeta))|_{\zeta = jve^{j\omega t}/2}. \tag{11}$$

With a little more work, this formula will give us the quadrature-measurement statistics when the oscillator is in a number state. That work, however, involves a digression into operator algebra.

If a and b are numbers, then $e^{a+b} = e^a e^b = e^b e^a$. If A and B are square matrices, then the matrix exponentials e^{A+B}, e^A, and e^B are defined by their respective Taylor series, i.e.,

$$e^C \equiv \sum_{n=0}^{\infty} \frac{C^n}{n!}, \quad \text{for } C = (A+B), A, B. \tag{12}$$

If A and B commute, then you can easily verify that $e^{A+B} = e^A e^B = e^B e^A$, but if they do *not* commute, then these equalities do *not* hold. Because our Hilbert space operators for the quantum harmonic oscillator are, in essence, infinite-dimensional square matrices—as explicitly represented by their number-ket matrix elements—we know that $e^{\hat{A}+\hat{B}}$, $e^{\hat{A}}e^{\hat{B}}$, and $e^{\hat{B}}e^{\hat{A}}$ are three *different* operators when $[\hat{A}, \hat{B}] \neq 0$. However, a special case of the Baker-Campbell-Hausdorff theorem, which you will use on Problem Set 5, is very helpful in this regard:[2] It states that if $\hat{A}$ and $\hat{B}$ are non-commuting operators that commute with their commutator, i.e., they satisfy

$$\left[\hat{A}, [\hat{A}, \hat{B}]\right] = \left[\hat{B}, [\hat{A}, \hat{B}]\right] = 0, \tag{13}$$

then

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A}, \hat{B}]/2} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]/2}. \tag{14}$$

Let's apply the preceding theorem to the Wigner characteristic function, by identifying $\hat{A} = -\zeta^* \hat{a}$ and $\hat{B} = \zeta \hat{a}^\dagger$. We then get

$$[\hat{A}, \hat{B}] = -|\zeta|^2 [\hat{a}, \hat{a}^\dagger] = -|\zeta|^2, \tag{15}$$

[2]For more information, see W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) Sect. 3.1, or M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000) Sect. 7.4.2.

so that the premise of the theorem is satisfied, whence

$$\chi_W(\zeta^*,\zeta) \equiv \langle e^{-\zeta^*\hat{a}+\zeta\hat{a}^\dagger}\rangle = \chi_A(\zeta^*,\zeta)e^{|\zeta|^2/2} = \chi_N(\zeta^*,\zeta)e^{-|\zeta|^2/2}. \tag{16}$$

Here, $\chi_A(\zeta^*,\zeta)$ and $\chi_N(\zeta^*,\zeta)$ are the anti-normally ordered and normally-ordered quantum characteristic functions,

$$\chi_A(\zeta^*,\zeta) \equiv \langle e^{-\zeta^*\hat{a}}e^{\zeta\hat{a}^\dagger}\rangle \quad \text{and} \quad \chi_N(\zeta^*,\zeta) \equiv \langle e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}\rangle, \tag{17}$$

respectively.[3]

Now we are ready to confront the quadrature-measurement statistics of the number state $|n\rangle$, where, for simplicity, we will will start by assuming $t=0$ so that $\hat{a}_1(0)$ becomes $\hat{a}_1$. We have that

$$M_{a_1}(jv) = \chi_W(\zeta^*,\zeta))|_{\zeta=jv/2} = [\chi_N(\zeta^*,\zeta)e^{-|\zeta|^2/2}]|_{\zeta=jv/2} \tag{18}$$

$$= [\langle n|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}|n\rangle e^{-|\zeta|^2/2}]|_{\zeta=jv/2}. \tag{19}$$

Expanding the exponentials in their Taylor series, we can use the number-ket representations of the annihilation and creation operators, and the orthonormality of the number kets to show that

$$M_{a_1}(jv) = \left[\left(\sum_{m=0}^{\infty}\frac{\zeta^m}{m!}\langle n|\hat{a}^{\dagger m}\right)\left(\sum_{k=0}^{\infty}\frac{(-\zeta)^{*k}}{k!}\hat{a}^k|n\rangle\right)e^{-|\zeta|^2/2}\right]_{\zeta=jv/2} \tag{20}$$

$$= \left(\sum_{m=0}^{n}\frac{(jv/2)^m}{m!}\sqrt{\frac{n!}{(n-m)!}}\langle n-m|\right)\left(\sum_{k=0}^{n}\frac{(jv/2)^k}{k!}\sqrt{\frac{n!}{(n-k)!}}|n-k\rangle\right)e^{-v^2/8} \tag{21}$$

$$= \left(\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{(-v^2/4)^m}{m!}\right)e^{-v^2/8} = L_n(v^2/4)e^{-v^2/8}, \tag{22}$$

where

$$L_n(z) \equiv \sum_{m=0}^{n}(-1)^m\frac{n!}{m!(n-m)!}\frac{z^m}{m!}, \tag{23}$$

[3]An operator that is a polynomial in $\hat{a}$ and $\hat{a}^\dagger$ is said to be in anti-normal order when, in each of its terms, all of the annihilation operators stand to the left of all the creation operators, e.g., $\hat{a}^2\hat{a}^\dagger$ is anti-normally ordered, but $\hat{a}\hat{a}^\dagger\hat{a}$ is not. Similarly, an operator that is a polynomial in $\hat{a}$ and $\hat{a}^\dagger$ is normally ordered when, in each of its terms, all of the creation operators stand to the left of all the annihilation operators. Because the operator exponentials appearing inside the averaging brackets in χ_A and χ_N are defined by their power series, it is easily seen that these characteristics functions are averages of anti-normally ordered and normally-ordered operators, respectively. Operator ordering is very important in quantum optics, and some of its features will be explored on Problem Set 5.

is the nth Laguerre polynomial. Invoking the inverse transform that relates $M_{a_1}(jv)$ to $p_{a_1}(\alpha_1)$ we find that

$$p_{a_1}(\alpha_1) = \int_{-\infty}^{\infty} \frac{\mathrm{d}v}{2\pi}\, L_n(v^2/4)e^{-v^2/8}e^{-jv\alpha_1} = \sqrt{\frac{2}{\pi}}\,\frac{e^{-2\alpha_1^2}}{2^n n!}\,[H_n(\sqrt{2}\alpha_1)]^2, \tag{24}$$

where

$$H_n(z) \equiv (-1)^n e^{z^2}\frac{\mathrm{d}^n e^{-z^2}}{\mathrm{d}z^n}, \tag{25}$$

is the nth Hermite polynomial. Now, because $\hat{a}(t) = \hat{a}e^{-j\omega t}$, you can easily verify that

$$\chi_W(\zeta^*, \zeta)|_{\zeta = jve^{j\omega t}/2} = L_n(v^2/4)e^{-v^2/8}, \quad \text{for all } t, \tag{26}$$

which implies that

$$p_{a_1(t)}(\alpha_1) = \sqrt{\frac{2}{\pi}}\,\frac{e^{-2\alpha_1^2}}{2^n n!}\,[H_n(\sqrt{2}\alpha_1)]^2, \quad \text{for all } t. \tag{27}$$

We can now provide the promised more explicit description of the phase space pictures that we have drawn—on Slides 4 and 5—for the coherent state, number state, and squeezed state cases. The shaded regions can be regarded as the high-probability density regions for these measurements. For coherent states and squeezed states the quadrature-measurements pdfs are Gaussians. Hence, their high probability regions are circles (for coherent states) or ellipses (for squeezed states) whose centers are at $\langle \hat{a} \rangle$. Equation (27) shows that the $\hat{a}_1(t)$ statistics for the number state are time independent. Because $\hat{a}_1(t) = \mathrm{Re}(\hat{a}e^{-j\omega t})$, this tells us that our phase space plot on Slide 4 is correct, in that it is circularly symmetric. Moreover, as we can see from the typical example shown on Slide 9, the pdf for the number state's quadrature measurement has oscillations, but peaks away from $\alpha_1 = 0$. Thus the donut shaped phase-space plot on Slide 4 is appropriate for capturing its highest probability region.

The Wigner Distribution and Non-Classicality

The measurement statistics of $\hat{a}_1(t)$, for all t, follow from knowledge of the Wigner characteristic function $\chi_W(\zeta^*, \zeta) \equiv \langle e^{-\zeta^*\hat{a}+\zeta\hat{a}^\dagger} \rangle$. You should verify that the measurement statistics of $\hat{a}_2(t)$, for all t, also follow from knowledge of the Wigner characteristic function. So, it would seem that there is information about both quadratures in this characteristic function. Note that χ_W is a function of *two* real variables, ζ_1 and ζ_2, the real and imaginary parts of ζ, but the classical characteristic function for a quadrature measurement—and likewise its associated probability density function—only depends on *one* real variable. The classical characteristic function for the quadrature measurement and its associated pdf are a Fourier transform pair. It is reasonable, then, to inquire about what information might be had if we investigated the 2-D Fourier partner of χ_W. In particular, this partner might tell us how to get simultaneous information about both quadratures by some yet to be described measurement technique.

The 2-D Fourier partner of χ_W is the Wigner distribution, and it is defined as follows,

$$W(\alpha^*,\alpha) \equiv \int \frac{\mathrm{d}^2\zeta}{\pi^2}\,\chi_W(\zeta^*,\zeta)e^{\zeta^*\alpha-\zeta\alpha^*}, \tag{28}$$

where

$$\int \frac{\mathrm{d}^2\zeta}{\pi^2} \equiv \int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta_1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta_2}{\pi}, \tag{29}$$

and α is complex valued with real and imaginary parts α_1 and α_2, respectively. Because $\zeta^*\alpha - \zeta\alpha^* = 2j\zeta_1\alpha_2 - 2j\zeta_2\alpha_1$, (28) is, in essence, a Fourier transform relation. The inverse relation corresponding to this result is

$$\chi_W(\zeta^*,\zeta) = \int \mathrm{d}^2\alpha\, W(\alpha^*,\alpha)e^{-\zeta^*\alpha+\zeta\alpha^*}, \tag{30}$$

In classical probability theory, the inverse transform of a characteristic function is a probability density function. So, were the Wigner characteristic function to be the joint characteristic function for *two* quantum measurements—because, after all, it is a function of *two* real variables—its inverse transform, $W(\alpha^*,\alpha)$ would be a joint probability density function for two real-valued random variables. Evaluating Eq. (30) at $\zeta = 0$ gives

$$\int \mathrm{d}^2\alpha\, W(\alpha^*,\alpha) = 1, \tag{31}$$

because $\chi_W(0,0) = 1$, by definition. This is one of the two conditions that $W(\alpha^*,\alpha)$ must satisfy in order for it to be a 2-D probability density function. The following pair of examples will reveal that the other condition, $W(\alpha^*,\alpha) \geq 0$, need *not* be satisfied.

Example 1: the coherent state

When the oscillator is in the coherent state $|\beta\rangle$, we have that

$$\chi_W(\zeta^*,\zeta) = \langle\beta|e^{\zeta\hat{a}^\dagger}e^{-\zeta^*\hat{a}}|\beta\rangle e^{-|\zeta|^2/2} = e^{\zeta\beta^*-\zeta^*\beta}e^{-|\zeta|^2/2}, \tag{32}$$

which, in turn, yields,

$$W(\alpha^*,\alpha) = \frac{e^{-2|\alpha-\beta|^2}}{\pi/2}. \tag{33}$$

This last result says that $W(\alpha^*,\alpha)$ is the joint probability density function for statistically independent, Gaussian random variables α_1 and α_2, whose mean values are β_1 and β_2, and whose variances are both 1/4. We have already stated that the coherent states represent the classical behavior of the oscillator, through the correspondence principle, so it should not be too surprising that their Wigner distribution functions behave like classical probability densities.

Example 2: the photon number state
When the oscillator's state is the number ket $|n\rangle$, the calculation we did for the number state's quadrature-measurement statistics gives us that

$$W(\alpha^*, \alpha) = \int \frac{\mathrm{d}^2\zeta}{\pi^2} L_n(|\zeta|^2) e^{-|\zeta|^2/2} e^{\zeta^*\alpha - \zeta\alpha^*}. \tag{34}$$

Converting the 2-D integral to polar coordinates, using $\zeta = re^{j\phi}$ and $\alpha = |\alpha|e^{j\theta}$, and exploiting $\zeta^*\alpha - \zeta\alpha^* = -2jr|\alpha|\sin(\phi - \theta)$, we can perform the azimuthal-angle integration and obtain

$$W(\alpha^*, \alpha) = \frac{2}{\pi}\int_0^\infty \mathrm{d}r\, r L_n(r^2) e^{-r^2/2} J_0(2r|\alpha|) = (-1)^n \frac{2}{\pi} L_n(4|\alpha|^2) e^{-2|\alpha|^2}, \tag{35}$$

where J_0 is the zero-th order Bessel function of the first kind. When $n = 0$, this reduces to the zero-mean, coherent state result,

$$W(\alpha^*, \alpha) = \frac{2}{\pi} e^{-2|\alpha|^2}, \tag{36}$$

because $L_0(z) = 1$. However, when $n = 1$, we get

$$W(\alpha^*, \alpha) = \frac{2}{\pi}(4|\alpha|^2 - 1) e^{-2|\alpha|^2}, \tag{37}$$

because $L_1(z) = (1 - z^2)$. This $W(\alpha^*, \alpha)$ *cannot* be a classical probability density function, because it is negative for $|\alpha| < 1/2$.

Any state whose Wigner distribution is not a classical probability density function is a *non-classical state*. This non-classicality condition is *not* sufficiently general, however. In particular, as we shall see later this term, the squeezed states are non-classical, even though their Wigner distribution functions are 2-D classical probability densities.

Positive operator-Valued Measurement of $\hat{a}$

We failed in our attempt to exploit the Wigner distribution function as a means for "measuring" both quadratures of the oscillator at once, because number states with $n \neq 0$ yield Wigner distributions that are negative for some α values. However, there *is* a way to get information about $\hat{a}_1$ and $\hat{a}_2$ at the same time. To do so, of course, we must go beyond the realm of the oscillator's observables, because we know that non-commuting observables—like $\hat{a}_1$ and $\hat{a}_2$—cannot be measured simultaneously. The key to getting some information about both quadratures is the overcompleteness property of the coherent states, as we will now show.

The Coherent State Positive Operator-Valued Measurement
We shall *define* the POVM for the annihilation operator $\hat{a}$ as follows. The outcome

of this measurement is a complex number, α, with real and imaginary parts α_1 and α_2, respectively. The probability density for getting the complex-valued outcome α—equivalently, the joint probability density for getting the two real-valued outcomes α_1 and α_2—when the oscillator's state is $|\psi\rangle$ is[4]

$$p(\alpha) = \frac{|\langle\alpha|\psi\rangle|^2}{\pi}, \quad \text{for } \alpha \in \mathcal{C}. \tag{38}$$

We can easily show that this definition is consistent with classical probability theory. It is clear that $p(\alpha)$ so defined cannot be negative. To prove that it integrates to one, we use the overcompleteness of the coherent states to write

$$\int d^2\alpha\, p(\alpha) = \int \frac{d^2\alpha}{\pi} \langle\psi|\alpha\rangle\langle\alpha|\psi\rangle = \langle\psi| \left(\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|\right) |\psi\rangle = \langle\psi|\hat{I}|\psi\rangle = 1. \tag{39}$$

In our next lecture, we will reconcile this POVM with the notion that measurements should be observables. For the rest of today, however, we shall evaluate some examples.

Example 1: the photon number state
When the oscillator's state is the number ket $|n\rangle$, we have that

$$p(\alpha) = \frac{|\langle\alpha|n\rangle|^2}{\pi} = \frac{|\alpha|^{2n}e^{-|\alpha|^2}}{\pi n!}, \quad \text{for } \alpha \in \mathcal{C}. \tag{40}$$

Although this *looks* like a Poisson distribution, it is not. This is a probability density function on α with n as a parameter. It is *not* a probability mass function for n with α as a parameter. If we write $\alpha = |\alpha|e^{j\theta}$, we see that $|\alpha|$ and θ are statistically independent, with the former having the pdf

$$p(|\alpha|) = \frac{2|\alpha|^{2n}e^{-|\alpha|^2}}{n!}, \quad \text{for } 0 \le |\alpha| < \infty, \tag{41}$$

and the latter being uniformly distributed on $[0, 2\pi]$. Polar coordinate evaluation then makes it easy to show that the $\hat{a}$ measurement has the following mean value,

$$\langle\alpha\rangle = \int d^2\alpha\, \alpha p(\alpha) = \int_0^\infty d|\alpha|\, \frac{2|\alpha|^{2(n+1)}e^{-|\alpha|^2}}{n!} \int_0^{2\pi} d\theta\, \frac{e^{i\theta}}{2\pi} = 0, \tag{42}$$

and that its real and imaginary parts have the following variances,

$$\langle\Delta\alpha_1^2\rangle = \int d^2\alpha\, \alpha_1^2 p(\alpha) = \int_0^\infty d|\alpha|\, \frac{2|\alpha|^{2n+3}e^{-|\alpha|^2}}{n!} \int_0^{2\pi} d\theta\, \frac{\cos^2(\theta)}{2\pi} = \frac{n+1}{2} \tag{43}$$

$$\langle\Delta\alpha_2^2\rangle = \int d^2\alpha\, \alpha_2^2 p(\alpha) = \int_0^\infty d|\alpha|\, \frac{2|\alpha|^{2n+3}e^{-|\alpha|^2}}{n!} \int_0^{2\pi} d\theta\, \frac{\sin^2(\theta)}{2\pi} = \frac{n+1}{2}. \tag{44}$$

[4]We call this specification a measurement of $\hat{a}$ because the outcomes are eigenvalues of $\hat{a}$ and the probability density for getting a particular outcome is proportional to the squared magnitude of the projection of the system state onto the $\hat{a}$ eigenket that is associated with that outcome.

State	$\langle\alpha\rangle$
$\lvert n\rangle$	0
$\lvert\beta\rangle$	β
$\lvert\beta;\mu,\nu\rangle$	$\mu^*\beta-\nu\beta^*$

Table 3: Mean values of the $\hat{a}$ POVM for number states, coherent states, and squeezed states.

State	$\langle\Delta\alpha_1^2\rangle$	$\langle\Delta\alpha_2^2\rangle$
$\lvert n\rangle$	$(n+1)/2$	$(n+1)/2$
$\lvert\beta\rangle$	$1/2$	$1/2$
$\lvert\beta;\mu,\nu\rangle$	$(\lvert\mu-\nu\rvert^2+1)/4$	$(\lvert\mu+\nu\rvert^2+1)/4$

Table 4: Variances of the real and imaginary parts of the $\hat{a}$ POVM for number states, coherent states, and squeezed states.

Example 2: the coherent state
When the oscillator is in the coherent state $|\beta\rangle$, we have that

$$p(\alpha) = \frac{|\langle\alpha|\beta\rangle|^2}{\pi} = \frac{e^{-|\alpha-\beta|^2}}{\pi}, \tag{45}$$

so that the real and imaginary parts of the $\hat{a}$ measurement are statistically independent, Gaussian random variables with mean values β_1 and β_2, and identical variances of 1/2.

Example 3: the squeezed state
When the oscillator is in the squeezed state $|\beta;\mu,\nu\rangle$ its $\hat{a}$-measurement statistics could be derived, from the knowledge we already have, but we will postpone that derivation until later when an easier route becomes available. For now, we will just state that

$$\langle\alpha\rangle = \mu^*\beta - \nu\beta^*, \quad \langle\Delta\alpha_1^2\rangle = \frac{(|\mu-\nu|^2+1)}{4}, \quad \langle\Delta\alpha_2^2\rangle = \frac{(|\mu+\nu|^2+1)}{4}. \tag{46}$$

The preceding mean value and variance results are summarized in Tables 3 and 4, respectively.

Some discussion is very certainly in order at this point. It is evident, from Table 3, that for the states with non-zero $\langle\hat{a}\rangle$ values, the POVM specified by (38) gives *simultaneous* information about the mean values of *both* the $\hat{a}_1$ and $\hat{a}_2$ quadratures.

Table 4 shows that the Heisenberg uncertainty principle for the quadratures is *not* being violated by this $\hat{a}$ measurement. In particular, for number states, coherent states, and squeezed states we have that

$$\langle \Delta\alpha_1^2 \rangle \langle \Delta\alpha_2^2 \rangle \geq 1/4, \tag{47}$$

where the number state and coherent state results are self-evident, and the squeezed-state result follows from

$$|\mu - \nu|^2 |\mu + \nu|^2 = |\mu^2 - \nu^2|^2 \geq (|\mu|^2 - |\nu|^2)^2 = 1, \tag{48}$$

and

$$|\mu - \nu|^2 + |\mu + \nu|^2 = 2|\mu|^2 + 2|\nu|^2 = 2 + 4|\nu|^2 \geq 2. \tag{49}$$

Comparing the variances shown in Table 4 for the $\hat{a}$ measurement with our previous variance results for the $\hat{a}_1$ and $\hat{a}_2$ measurements from Table 2 shows that each of the former variances exceeds its corresponding latter result by an additive term of 1/4. Thus, for a coherent state we have $\langle \Delta\hat{a}_k^2 \rangle = 1/4$ and $\langle \Delta\alpha_k^2 \rangle = 1/2$, for $k = 1, 2$. Learning the physical origin of this extra noise will be quite important to us, but that lesson must wait until the next lecture.

The Road Ahead

We are almost done with our basic treatment of the quantum harmonic oscillator. In the next lecture we shall reconcile the $\hat{a}$-POVM with the notion of observables, and, in doing so, we shall identify the physical origin of the extra noise term of 1/4 in Table 4. We will then use our knowledge of the quantum harmonic oscillator to begin our study of single-mode photodetection.