6.453 Quantum Optical Communication
Spring 2009

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6.453 *Quantum Optical Communication*
Lecture 6

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6.453 *Quantum Optical Communication* — Lecture 6

- Announcements
  - Pick up problem set 2 graded, lecture notes, slides
- Quantum Harmonic Oscillator
  - Quadrature-representation wave functions
  - Minimum uncertainty-product (MUP) states
  - Squeezed states and their measurement statistics

Coherent States: Reprise

- Quantum Harmonic Oscillator:

$$\hat{a}(t) = \hat{a}_1(t) + j\hat{a}_2(t) = \hat{a}e^{-j\omega t}$$

- Heisenberg Uncertainty Principle:

$$\langle \Delta \hat{a}_1^2(t) \rangle \langle \Delta \hat{a}_2^2(t) \rangle \geq \frac{1}{16}$$

- Eigenkets of the Annihilation Operator:

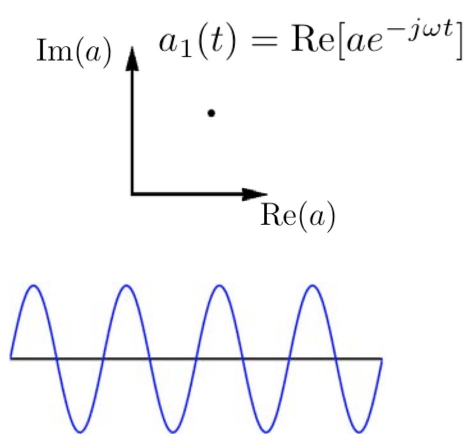
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad \text{for } \alpha = \alpha_1 + j\alpha_2$$

- Quadrature Measurement Statistics:

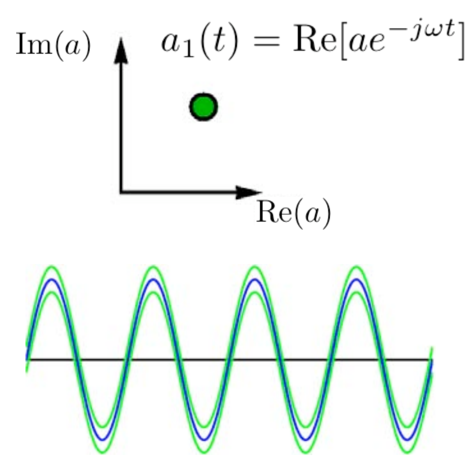
$$\langle \alpha|\hat{a}(t)|\alpha\rangle = \alpha e^{-j\omega t}, \quad \langle \alpha|\Delta\hat{a}_1^2(t)|\alpha\rangle = \langle \alpha|\Delta\hat{a}_2^2(t)|\alpha\rangle = \frac{1}{4}$$

Classical versus Quantum Quadrature Behavior

- Classical Oscillator: Noiseless



- Quantum Oscillator: $|\alpha\rangle$ State



The Vacuum State is Special!

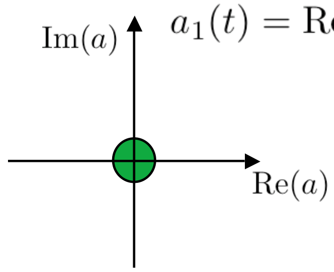
- It's a number state: $\hat{N}|0\rangle = 0$
- Zero-uncertainty number operator measurement outcome:

$$\Pr(\,\hat{N} = 0 \mid |0\rangle\,) = 1$$

- Zero-point energy:

$$\Pr(\,\hat{H} = \hbar\omega/2 \mid |0\rangle\,) = 1$$

$$a_1(t) = \mathrm{Re}[ae^{-j\omega t}]$$



- It's a coherent state: $\hat{a}|0\rangle = 0$
- Minimum uncertainty-product quadrature measurements

$$\langle 0|\Delta\hat{a}_1^2(t)|0\rangle\langle 0|\Delta\hat{a}_2^2(t)|0\rangle = \frac{1}{16}$$

- Zero-point fluctuations:

$$\langle 0|\Delta\hat{a}_1^2(t)|0\rangle = \langle 0|\Delta\hat{a}_2^2(t)|0\rangle = \frac{1}{4}$$

Quadrature-Operator Eigenkets

- On Problem Set 4 You Will Find Quadrature Eigenkets:

$$\hat{a}_1|\alpha_1\rangle_1 = \alpha_1|\alpha_1\rangle_1 \quad \text{and} \quad \hat{a}_2|\alpha_2\rangle_2 = \alpha_2|\alpha_2\rangle_2$$

  - Orthonormal:

$${}_1\langle\alpha_1|\beta_1\rangle_1 = \delta(\alpha_1 - \beta_1) \quad \text{and} \quad {}_2\langle\alpha_2|\beta_2\rangle_2 = \delta(\alpha_2 - \beta_2)$$

  - Complete:

$$\hat{I} = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, |\alpha_1\rangle_{1\,1}\langle\alpha_1| = \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, |\alpha_2\rangle_{2\,2}\langle\alpha_2|$$

  - Fourier-kernel relation:

$${}_2\langle\alpha_2|\alpha_1\rangle_1 = \frac{e^{-2j\alpha_2\alpha_1}}{\sqrt{\pi}}$$

Quadrature Representations: Wave Functions

- Wave Function Representations of Arbitrary Ket

$$|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \psi(\alpha_1)|\alpha_1\rangle_1 = \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, \Psi(\alpha_2)|\alpha_2\rangle_2$$

- Fourier-Transform Duality

$$\begin{aligned}\Psi(\alpha_2) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \psi(\alpha_1)e^{-2j\alpha_2\alpha_1} \\ \psi(\alpha_1) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, \Psi(\alpha_2)e^{2j\alpha_1\alpha_2}\end{aligned}$$

MUP States with Zero-Mean Quadratures

- Quadrature Variances From Wave Functions

$$\langle\psi|\Delta\hat{a}_1^2|\psi\rangle = \langle\psi|\hat{a}_1^2|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \alpha_1^2|\psi(\alpha_1)|^2$$

$$\langle\psi|\Delta\hat{a}_2^2|\psi\rangle = \langle\psi|\hat{a}_2^2|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, \alpha_2^2|\Psi(\alpha_2)|^2$$

- Minimum Uncertainty-Product Wave Functions

$$\begin{aligned}\psi(\alpha_1) &= \frac{\exp(-\alpha_1^2/4\langle\Delta\hat{a}_1^2\rangle)}{(2\pi\langle\Delta\hat{a}_1^2\rangle)^{1/4}} \\ \Psi(\alpha_2) &= \frac{\exp(-4\langle\Delta\hat{a}_1^2\rangle\alpha_2^2)}{(\pi/8\langle\Delta\hat{a}_1^2\rangle)^{1/4}} = \frac{\exp(-\alpha_2^2/4\langle\Delta\hat{a}_2^2\rangle)}{(2\pi\langle\Delta\hat{a}_2^2\rangle)^{1/4}}\end{aligned}$$

Minimum Uncertainty-Product States

- Minimum Uncertainty-Product Wave Functions

$$\psi(\alpha_1) = \frac{\exp[2j\langle\hat{a}_2\rangle\alpha_1 - j\langle\hat{a}_1\rangle\langle\hat{a}_2\rangle - (\alpha_1 - \langle\hat{a}_1\rangle)^2/4\langle\Delta\hat{a}_1^2\rangle]}{(2\pi\langle\Delta\hat{a}_1^2\rangle)^{1/4}}$$

$$\Psi(\alpha_2) = \frac{\exp[-2j\langle\hat{a}_1\rangle\alpha_2 + j\langle\hat{a}_1\rangle\langle\hat{a}_2\rangle - (\alpha_2 - \langle\hat{a}_2\rangle)^2/4\langle\Delta\hat{a}_2^2\rangle]}{(2\pi\langle\Delta\hat{a}_2^2\rangle)^{1/4}}$$

where $\langle\Delta\hat{a}_1^2\rangle\langle\Delta\hat{a}_2^2\rangle = \frac{1}{16}$

- Coherent-State Wave Functions

$$\langle\Delta\hat{a}_1^2\rangle = \langle\Delta\hat{a}_2^2\rangle = \frac{1}{4}$$

Squeezed States: Connection to MUP States

- Equality Condition for Quadratures Uncertainty Principle:

$$\Delta\hat{a}_1|\psi\rangle = -j\lambda\Delta\hat{a}_2|\psi\rangle, \quad \lambda \text{ real}$$

- Equivalent to Bogoliubov-Transformation Eigenket:

$$\hat{b}|\beta;\mu,\nu\rangle \equiv (\mu\hat{a} + \nu\hat{a}^\dagger)|\beta;\mu,\nu\rangle = \beta|\beta;\mu,\nu\rangle$$

where μ, ν real and $\mu^2 - \nu^2 = 1$

Squeezed States: Dynamics

- Bogoliubov Transformation with μ, ν complex

$$\hat{b}|\beta;\mu,\nu\rangle \equiv (\mu\hat{a} + \nu\hat{a}^\dagger)|\beta;\mu,\nu\rangle = \beta|\beta;\mu,\nu\rangle$$

where $|\mu|^2 - |\nu|^2 = 1$

- Quadrature-Measurement Mean Values and Variances

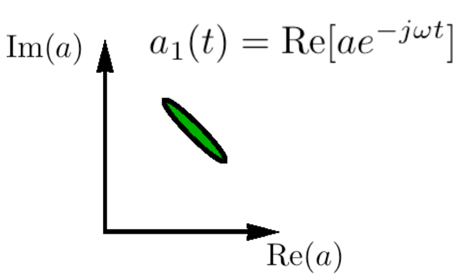
$$\langle\beta;\mu,\nu|\hat{a}(t)|\beta;\mu,\nu\rangle = (\mu^*\beta - \nu\beta^*)e^{-j\omega t}$$

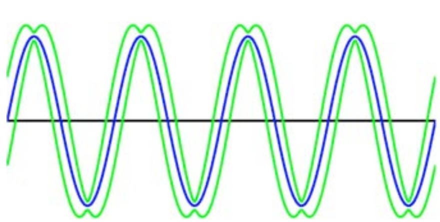
$$\langle\beta;\mu,\nu|\Delta\hat{a}_1^2(t)|\beta;\mu,\nu\rangle = \frac{1}{4}|\mu - \nu e^{-2j\omega t}|^2$$

$$\langle\beta;\mu,\nu|\Delta\hat{a}_2^2(t)|\beta;\mu,\nu\rangle = \frac{1}{4}|\mu + \nu e^{-2j\omega t}|^2$$

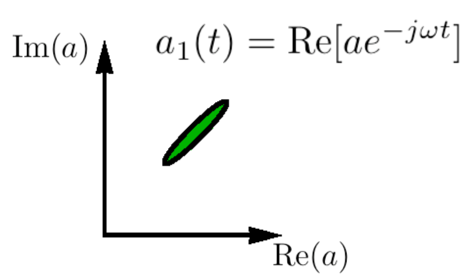
Quadrature-Measurement Statistics

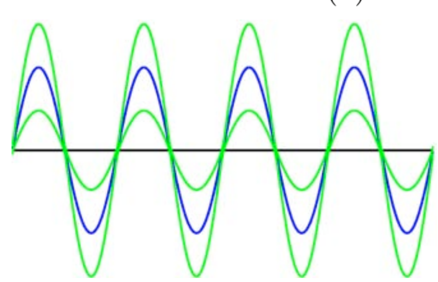
- Amplitude-Squeezed State





- Phase-Squeezed State





Coming Attractions: Lectures 7 and 8

- Lecture 7:
  Quantum Harmonic Oscillator
  - Wigner distribution and phase space
  - Probability operator-valued measurement of $\hat{a}$

- Lecture 8:
  Single-Mode Photodetection
  - Direct Detection
  - Homodyne Detection