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6.453 Quantum Optical Communication

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Reading: For squeezed states:

- C.C. Gerry and P.L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005) Sect. 7.1.
- H.P. Yuen, "Two-photon coherent states of the radiation field," Phys. Rev. A **13**, 2226-2243 (1976).
- L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995) Sects. 21.1–21.6. (On reserve at Barker library.)

For continuous-spectrum eigenkets:

- W.H. Louisell, *Quantum Statistical Properties of Radiation* (McGraw-Hill, New York, 1973) Sect. 1.10.

Introduction

Last time we showed that the annihilation operator, despite its not being Hermitian, has an overcomplete set of eigenkets—the coherent states $\{|\alpha\rangle : \alpha \in \mathcal{C}\}$—and that in the limit $|\alpha| \to \infty$ these states give the classical limit for the quantum harmonic oscillator. In particular, for the coherent state $|\alpha\rangle$ we have the following quantum measurement statistics

- Measurement of the number operator, $\hat{N}$, yields a classical random variable with mean value $|\alpha|^2$ and variance $|\alpha|^2$, so that the signal-to-noise ratio of the photon number measurement is

$$\text{SNR}_{\text{number}} \equiv \frac{\langle \hat{N}\rangle^2}{\langle \Delta\hat{N}^2\rangle} = |\alpha|^2 \to \infty, \quad \text{as } |\alpha| \to \infty. \tag{1}$$

The Chebyschev inequality then implies that the probability that the $\hat{N}$-measurement outcome deviates from $|\alpha|^2$ by $\epsilon|\alpha|^2$ will go to zero, as $|\alpha| \to \infty$, for all $\epsilon > 0$.

- Measurement of the quadrature operator, $\hat{a}_1(t)$, yields a classical random variable with mean value $\text{Re}(\alpha e^{-j\omega t})$, and variance $1/4$, so that the signal-to-noise ratio of the quadrature operator measurement is

$$\text{SNR}_{\text{quad}} \equiv \frac{\langle \hat{a}_1(t)\rangle^2}{\langle \Delta\hat{a}_1^2(t)\rangle} = 4[\text{Re}(\alpha(e^{-j\omega t})]^2 \to \infty, \quad \text{for } |\text{Re}(\alpha e^{-j\omega t})| \to \infty. \tag{2}$$

  Here, the Chebyschev inequality leads to the conclusion that quadrature measurement behavior is essentially noiseless sinusoidal oscillation in the limit of $|\alpha| \to \infty$.[1]

The variance-1/4 quadrature fluctuations,

$$\langle \Delta\hat{a}_1^2(t)\rangle = \langle \Delta\hat{a}_2^2(t)\rangle = 1/4, \tag{3}$$

satisfy the Heisenberg uncertainty principle with equality, and hence the coherent states are minimum uncertainty-product (MUP) states for the inequality

$$\langle \Delta\hat{a}_1^2(t)\rangle\langle \Delta\hat{a}_2^2(t)\rangle \geq 1/16. \tag{4}$$

An especially interesting special case is then the coherent state with zero eigenvalue, $|0\rangle$, because this state is also a photon number eigenket, and hence an energy eigenket, viz.,

$$\hat{a}|0\rangle = 0, \quad \hat{N}|0\rangle = 0, \quad \hat{H}|0\rangle = \frac{\hbar\omega}{2}\,|0\rangle. \tag{5}$$

The last of these three eigenket-eigenvalue relations shows that the energy in the state $|0\rangle$ is $\hbar\omega/2$, the zero-point energy. The second of these three eigenket-eigenvalue relations shows that there are zero energy quanta (zero photons) in the state $|0\rangle$, so it is appropriate and conventional to call this the *vacuum state*, because it represents an unexcited state of the oscillator. However, because $|0\rangle$ is a coherent state, its quadrature variances satisfy (3). The noise that is quantified by these variances is called *zero-point fluctuations*, because it is associated with the zero-point energy in the vacuum state. That zero-point energy leads to a deterministic non-zero result when the Hamiltonian $\hat{H}$ measurement is performed on the vacuum state, whereas a zero-mean variance-1/4 random variable results when a quadrature operator—$\hat{a}_1(t)$ or $\hat{a}_2(t)$—measurement is performed on this state is a clear manifestation of something that we noted earlier: the state of a quantum system *and* the measurement that is made determine the statistics of the resulting outcome.

Today's lecture will be devoted to learning more about MUP states for the Heisenberg inequality (4). We begin by developing the minimum uncertainty-product property from the perspective of wave functions. We'll then connect this formulation back to what we have already seen, from our Dirac-notation treatment of the coherent states, and then move on to introduce the squeezed states, which are MUP states with *unequal* fluctuations in the two quadratures, i.e., these states satisfy

$$\langle \Delta\hat{a}_1^2(t)\rangle\langle \Delta\hat{a}_2^2(t)\rangle = 1/16 \quad \text{with} \quad \langle \Delta\hat{a}_1^2(t)\rangle \neq \langle \Delta\hat{a}_2^2(t)\rangle. \tag{6}$$

[1]The same behavior also applies to the other quadrature, $\hat{a}_2(t)$.

Quadrature-Operator Measurements and Wave Functions

On Problem Set 4 you will determine key properties of the quadrature-operator eigenkets. Specifically, for $\hat{a}_1 = \mathrm{Re}(\hat{a})$ and $\hat{a}_2 = \mathrm{Im}(\hat{a})$, with $\hat{a} = \hat{a}(0)$, these eigenkets are as follows:

$$\hat{a}_k|\alpha_k\rangle_k = \alpha_k|\alpha_k\rangle_k, \quad \text{for } k = 1, 2, \tag{7}$$

where the eigenvalues span the continuum $-\infty < \alpha_k < \infty$, and the k subscript on the right-angle bracket in the ket is to indicate for which quadrature this state is an eigenket. From Axiom 3a, we know that these eigenkets are infinite length, satisfying the delta-function orthonormality relation,

$${}_k\langle\alpha_k|\beta_k\rangle_k = \delta(\alpha_k - \beta_k), \quad \text{for } k = 1, 2. \tag{8}$$

We also know that they resolve the identity according to

$$\hat{I} = \int_{-\infty}^{\infty} \mathrm{d}\alpha_k \, |\alpha_k\rangle_{kk}\langle\alpha_k|, \quad \text{for } k = 1, 2, \tag{9}$$

and diagonalize their associated quadrature operators,

$$\hat{a}_k = \int_{-\infty}^{\infty} \mathrm{d}\alpha_k \, \alpha_k|\alpha_k\rangle_{kk}\langle\alpha_k|, \quad \text{for } k = 1, 2. \tag{10}$$

Furthermore, if the oscillator is in the (finite-energy $\leftrightarrow$ unit-length) state $|\psi\rangle$, then measurement of the quadrature operator $\hat{a}_k$ yields a continuous, classical random variable whose probability density function is[2]

$$p(\alpha_k) = |{}_k\langle\alpha_k|\psi\rangle|^2, \quad \text{for } k = 1, 2. \tag{11}$$

The final result that we need from Problem Set 4 is the inner product relation that links $|\alpha_1\rangle_1$ to $|\alpha_2\rangle_2$:

$${}_2\langle\alpha_2|\alpha_1\rangle_1 = \frac{e^{-2j\alpha_2\alpha_1}}{\sqrt{\pi}}. \tag{12}$$

First treatments of quantum mechanics often represent states as *wave functions*. We can now connect our Dirac-notation work to that approach. Let $|\psi\rangle$ be some arbitrary (unit-length) state. Then $|\psi\rangle$ has two equivalent representation—which follow from resolving the identity using the $|\alpha_1\rangle_1$ and $|\alpha_2\rangle_2$ kets, respectively—namely,

$$|\psi\rangle = \hat{I}|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_k \, ({}_k\langle\alpha_k|\psi\rangle)|\alpha_k\rangle_k, \quad \text{for } k = 1, 2. \tag{13}$$

[2]Because $|\alpha_k\rangle_k$ has infinite length, the projection postulate clearly does *not* apply to quadrature measurements. Later this term, when we see how this measurement is realized via optical homodyne detection, you will become more comfortable with abandoning the projection postulate for this case.

So, with

$$\psi(\alpha_1) \equiv {}_1\langle\alpha_1|\psi\rangle \quad \text{and} \quad \Psi(\alpha_2) \equiv {}_2\langle\alpha_2|\psi\rangle, \tag{14}$$

we have that

$$|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \psi(\alpha_1)|\alpha_1\rangle_1 \quad \text{and} \quad |\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, \Psi(\alpha_2)|\alpha_2\rangle_2, \tag{15}$$

where the *wave functions*, $\{\psi(\alpha_1), \Psi(\alpha_2)\}$, have squared magnitudes that integrate to unity over the infinite interval $-\infty < \alpha_k < \infty$, for $k = 1, 2$, respectively.[3] In terms of these wave functions, the quadrature measurement statistics for $\hat{a}_1$ and $\hat{a}_2$ become the following. If the state of the system is $|\psi\rangle$, we get a classical random variable with probability density

$$p(\alpha_1) = |\psi(\alpha_1)|^2, \tag{16}$$

when we measure $\hat{a}_1$, and we get a classical random variable with probability density

$$p(\alpha_2) = |\Psi(\alpha_2)|^2, \tag{17}$$

when we measure $\hat{a}_2$. Now, the inner product relation between the $\hat{a}_1$ and $\hat{a}_2$ eigenkets implies the following relations between the wave functions, $\{\psi(\alpha_1), \Psi(\alpha_2)\}$, for the state $|\psi\rangle$:

$$\Psi(\alpha_2) = {}_2\langle\alpha_2|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, {}_2\langle\alpha_2|\alpha_1\rangle_1 \psi(\alpha_1) \tag{18}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \psi(\alpha_1) \frac{e^{-2j\alpha_2\alpha_1}}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\, \mathcal{F}[\psi(t)]\big|_{f=\alpha_2/\pi}\,, \tag{19}$$

and

$$\psi(\alpha_1) = {}_1\langle\alpha_1|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, {}_1\langle\alpha_1|\alpha_2\rangle_2 \Psi(\alpha_2) \tag{20}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, \Psi(\alpha_2) \frac{e^{2j\alpha_2\alpha_1}}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}\, \mathcal{F}^{-1}[\Psi(f)]\big|_{t=\alpha_1/\pi}\,. \tag{21}$$

Here,

$$\mathcal{F}[x(t)] \equiv \int_{-\infty}^{\infty} \mathrm{d}t\, x(t) e^{-j2\pi ft} \quad \text{and} \quad \mathcal{F}^{-1}[X(f)] \equiv \int_{-\infty}^{\infty} \mathrm{d}f\, X(f) e^{j2\pi ft}, \tag{22}$$

denote the Fourier transform and its inverse. Simply stated, the $\hat{a}_1$ and $\hat{a}_2$ wave functions for the state $|\psi\rangle$ comprise a Fourier transform pair. Applying the Fourier transform uncertainty principle that you proved on Problem Set 2 then tells us that

$$\langle\hat{a}_1^2\rangle\langle\hat{a}_2^2\rangle = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, |\alpha_1|^2|\psi(\alpha_1)|^2 \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, |\alpha_2|^2|\Psi(\alpha_2)|^2 \geq 1/16. \tag{23}$$

[3]Thus, these wave functions live in the Hilbert space $\mathcal{L}_2[-\infty, \infty]$.

If $\langle \hat{a} \rangle = 0$, i.e., if the quadrature measurements are both zero-mean random variables, then the mean-squares that appear in this inequality become variances, and we regain the Heisenberg uncertainty principle for the quadratures that we originally derived in Dirac notation.[4]

Suppose, for now that $|\psi\rangle$ does have zero-mean quadrature measurements. Then, from our work on Problem Set 2, we know that equality will occur in the wave-function form of the Heisenberg uncertainty principle for the quadratures when the wave functions associated with $|\psi\rangle$ are the Gaussian-function Fourier pair

$$\psi(\alpha_1) = \frac{e^{-\alpha_1^2/4\langle \Delta \hat{a}_1^2 \rangle}}{(2\pi \langle \Delta \hat{a}_1^2 \rangle)^{1/4}} \quad \text{and} \quad \Psi(\alpha_2) = \frac{e^{-\alpha_2^2/4\langle \Delta \hat{a}_2^2 \rangle}}{(2\pi \langle \Delta \hat{a}_2^2 \rangle)^{1/4}}, \tag{24}$$

with $\langle \Delta \hat{a}_2^2 \rangle = 1/16 \langle \Delta \hat{a}_1^2 \rangle$.

Minimum Uncertainty-Product States

The first time you were taught about the Heisenberg uncertainty principle—whether in qualitative terms in high school, or more quantitatively in undergraduate physics—it was probably in the context of position and momentum, not the quadrature components of the photon annihilation operator. Nevertheless, you were surely told that you could get a state with very low uncertainty in position, but this came at the expense of a very large uncertainty in momentum. Conversely, so the story goes, you could get a state with a very low uncertainty in momentum, but it carried a very large uncertainty in position. The same is evidently true for the zero-mean MUP state that we found at the end of the last section: we can have a very low $\langle \Delta \hat{a}_1^2 \rangle$ but it comes with a very large $\langle \Delta \hat{a}_2^2 \rangle$, and, conversely, a very low $\langle \Delta \hat{a}_2^2 \rangle$ carries with it a very large $\langle \Delta \hat{a}_1^2 \rangle$. So, the coherent states are rather special, because they are MUP states with *equal* uncertainties in the two quadratures. Before we can find their wave function representations, however, we must generalize the work of the preceding section to allow for non-zero quadrature-measurement mean values. Here, the basic idea is obvious from Fourier theory, and the form taken by the probability densities for the $\hat{a}_1$ and $\hat{a}_2$ quadrature measurements.

Suppose we have a state whose $\hat{a}_1$ measurement statistics—governed by its $\psi(\alpha_1)$ wave function—are zero mean and variance σ_1^2:

$$\langle \hat{a}_1 \rangle = 0 \quad \text{and} \quad \langle \hat{a}_1^2 \rangle = \sigma_1^2. \tag{25}$$

Another state, whose wave function is $\psi(\alpha_1 - m_1)$ will have then have

$$\langle \hat{a}_1 \rangle = m_1 \quad \text{and} \quad \langle \Delta \hat{a}_1^2 \rangle = \langle (\hat{a}_1 - m_1)^2 \rangle = \sigma_1^2, \tag{26}$$

[4]If we had wanted to complicate the notation on Problem Set 2, we could have derived the Fourier transform uncertainty principle in a form that would have led our wave function result to immediately yield (4). Instead, we have chosen to take that step in this lecture.

as you should verify by checking that $\psi(\alpha_1 - m_1)$ is a valid wave function for a unit-length ket, and using the probability density function for the $\hat{a}_1$ measurement to relate the mean and variance for this shifted wave-function state to the corresponding quantities for the original state. Because a shift in the time domain corresponds to a phase shift in the frequency domain, the Fourier transform relation between $\psi(\alpha_1)$ and $\Psi(\alpha_2)$ tells us how the $\hat{a}_2$ wave function of a state is affected when that state's $\hat{a}_1$ wave function has been shifted. Obviously, we can do the same thing if we start from a state whose $\hat{a}_2$ measurement statistics—governed by its $\Psi(\alpha_2)$ wave function—are zero mean and variance σ_2^2: the state $\Psi(\alpha_2 - m_2)$ will have

$$\langle \hat{a}_2 \rangle = m_2 \quad \text{and} \quad \langle \Delta\hat{a}_2^2 \rangle = \langle (\hat{a}_2 - m_2)^2 \rangle = \sigma_2^2, \tag{27}$$

and its $\psi(\alpha_1)$ wave function will undergo a phase shift. With these concepts in hand—and after some tedious algebra—we have that

$$\psi(\alpha_1) = \frac{\exp[2j\langle \hat{a}_2\rangle\alpha_1 - j\langle \hat{a}_1\rangle\langle \hat{a}_2\rangle - (\alpha_1 - \langle \hat{a}_1\rangle)^2/4\langle \Delta\hat{a}_1^2\rangle]}{(2\pi\langle \Delta\hat{a}_1^2\rangle)^{1/4}} \tag{28}$$

$$\Psi(\alpha_2) = \frac{\exp[-2j\langle \hat{a}_1\rangle\alpha_2 + j\langle \hat{a}_1\rangle\langle \hat{a}_2\rangle - (\alpha_2 - \langle \hat{a}_2\rangle)^2/4\langle \Delta\hat{a}_2^2\rangle]}{(2\pi\langle \Delta\hat{a}_2^2\rangle)^{1/4}}, \tag{29}$$

are the wave functions, in the $\hat{a}_1$ and $\hat{a}_2$ eigenket representations, of a minimum uncertainty-product state whose quadrature-measurement means are $\{\langle \hat{a}_1\rangle, \langle \hat{a}_2\rangle\}$, whose quadrature-measurement variances are $\{\langle \Delta\hat{a}_1^2\rangle, \langle \Delta\hat{a}_2^2\rangle\}$, with the former being arbitrary real numbers and the latter being positive numbers obeying $\langle \Delta\hat{a}_1^2\rangle = 1/16\langle \Delta\hat{a}_2^2\rangle$.

We can now obtain the wave function representations of the coherent states. Let $|\beta\rangle$ be a coherent state, so that $\hat{a}|\beta\rangle = \beta|\beta\rangle$. We know that $\hat{a}_1$ and $\hat{a}_2$ measurements made on this state lead to the following first and second moments:

$$\langle \hat{a}_1\rangle = \beta_1, \quad \langle \hat{a}_2\rangle = \beta_2, \quad \langle \Delta\hat{a}_1^2\rangle = \langle \Delta\hat{a}_2^2\rangle = 1/4. \tag{30}$$

It immediately follows, from our general wave function results for an MUP state, that

$$\psi(\alpha_1) = \frac{\exp[2j\beta_2\alpha_1 - j\beta_1\beta_2 - (\alpha_1 - \beta_1)^2]}{(\pi/2)^{1/4}} \tag{31}$$

$$\Psi(\alpha_2) = \frac{\exp[-2j\beta_1\alpha_2 + j\beta_1\beta_2 - (\alpha_2 - \beta_2)^2]}{(\pi/2)^{1/4}}, \tag{32}$$

are its wave functions. Note that deriving these wave functions has been more than a sterile reprise of the minimum uncertainty-product property of the coherent states. Because $|\psi(\alpha_1)|^2$ and $|\Psi(\alpha_2)|^2$ are the probability densities for the $\hat{a}_1$ and $\hat{a}_2$ quadrature measurements, we learn that a coherent state has Gaussian probability densities for its quadrature-measurement statistics. Indeed, for our general MUP state—which can have unequal variances in its two quadratures—the quadrature-measurement probability densities are Gaussian. Moreover, from our basic probability theory we know that these densities are completely characterized by their means and variances.

Let's take another look at minimum uncertainty-product states for the quadrature measurements $\hat{a}_1$ and $\hat{a}_2$, this time connecting back to the equality condition that we established when we derived the Heisenberg uncertainty principle. There we showed that

$$\langle \Delta \hat{a}_1^2 \rangle \langle \Delta \hat{a}_2^2 \rangle = 1/16 \tag{33}$$

will prevail if and only if the state $|\psi\rangle$ satisfies

$$\Delta \hat{a}_1 |\psi\rangle = -j\lambda \Delta \hat{a}_2 |\psi\rangle, \quad \text{where } \lambda \text{ is a real-valued constant.} \tag{34}$$

Rearranging this condition, as shown below, we can reduce it to an eigenket-eigenvalue relation:

$$(\hat{a}_1 + j\lambda \hat{a}_2)|\psi\rangle = (\langle \hat{a}_1 \rangle + j\lambda \langle \hat{a}_2 \rangle)|\psi\rangle. \tag{35}$$

We are not assured that there are solutions to this eigenket-eigenvalue relation for all values of λ, but some *must* exist because we have already identified MUP states for the quadrature operators' Heisenberg uncertainty principle via our wave function analysis.

Suppose that λ is positive. Writing the quadrature operators in terms of the annihilation and creation operators, and dividing by $\sqrt{\lambda}$ then leads to

$$(\mu \hat{a} + \nu \hat{a}^\dagger)|\psi\rangle = (\mu \langle \hat{a} \rangle + \nu \langle \hat{a}^\dagger \rangle)|\psi\rangle, \quad \text{where } \mu \equiv \frac{1+\lambda}{2\sqrt{\lambda}} \text{ and } \nu \equiv \frac{1-\lambda}{2\sqrt{\lambda}}. \tag{36}$$

If we define a new operator by

$$\hat{b} \equiv \mu \hat{a} + \nu \hat{a}^\dagger, \tag{37}$$

then, because μ and ν are real-valued and satisfy $\mu^2 - \nu^2 = 1$, we have that

$$[\hat{b}, \hat{b}^\dagger] = (\mu \hat{a} + \nu \hat{a}^\dagger)(\mu \hat{a}^\dagger + \nu \hat{a}) - (\mu \hat{a}^\dagger + \nu \hat{a})(\mu \hat{a} + \nu \hat{a}^\dagger) = \mu^2 [\hat{a}, \hat{a}^\dagger] + \nu^2 [\hat{a}^\dagger, \hat{a}] = 1, \tag{38}$$

i.e., $\hat{b}$ and $\hat{b}^\dagger$ have the *same* commutator as the annihilation operator and its adjoint, the creation operator. The transformation from $\{\hat{a}, \hat{a}^\dagger\}$ to $\{\hat{b}, \hat{b}^\dagger\}$ is known as a Bogoliubov transformation, and what we have just seen is that it preserves commutator brackets. Commutator preservation is an essential issue for us, in that it will dictate the need for quantum noise injection whenever there is loss or amplification of an electromagnetic field. For now, however, it will provide us with another window into MUPs with unequal quadrature fluctuations, one which we will ultimately see is how such states have been generated for the electromagnetic field.

We know—from our wave function results—that there *are* kets that satisfy (36), and we will denote them $|\beta; \mu, \nu\rangle$, whence

$$\hat{b}|\beta; \mu, \nu\rangle = \beta |\beta; \mu, \nu\rangle, \quad \text{for } \beta \in \mathcal{C}. \tag{39}$$

Now, by using the inverse of the Bogoliubov transformation that defines $\hat{b}$, i.e.,

$$\hat{a} = \mu\hat{b} - \nu\hat{b}^\dagger, \tag{40}$$

we find that

$$\langle\beta;\mu,\nu|\hat{a}|\beta;\mu,\nu\rangle = \mu\beta - \nu\beta^*, \tag{41}$$

so that putting the oscillator into this state produces simple harmonic motion in the mean, viz.,

$$\langle\beta;\mu,\nu|\hat{a}(t)|\beta;\mu\nu\rangle = (\mu\beta - \nu\beta^*)e^{-j\omega t}, \tag{42}$$

but the eigenket-eigenvalue relation (39) tells us *much* more. Because $\hat{b}$ and $\hat{b}^\dagger$ have the same commutator as $\hat{a}$ and $\hat{a}^\dagger$, it follows that

$$\hat{N}_b \equiv \hat{b}^\dagger\hat{b} \tag{43}$$

behaves like a number operator, i.e., it has non-negative integer eigenvalues with associated eigenkets $\{|n;\mu,\nu\rangle\}$ that obey

$$\langle m;\mu,\nu|n;\mu,\nu\rangle = \delta_{mn}, \quad \hat{N}_b = \sum_{n=0}^{\infty} n|n;\mu,\nu\rangle, \quad \hat{I} = \sum_{n=0}^{\infty} |n;\mu,\nu\rangle\langle n;\mu,\nu|. \tag{44}$$

Note that we have carried along μ and ν as *parameters* of these number-like states. This is to emphasize that unless $\mu = 1$ and $\nu = 0$, these states are *not* the photon-number eigenkets. However, using the preceding properties of the $\{|n;\mu,\nu\rangle\}$, we can show that

$$\hat{b} = \sum_{n=1}^{\infty} \sqrt{n}\,|n-1;\mu,\nu\rangle\langle n;\mu,\nu| \quad \text{and} \quad \hat{b}^\dagger = \sum_{n=0}^{\infty} \sqrt{n+1}\,|n+1;\mu,\nu\rangle\langle n;\mu,\nu|, \tag{45}$$

and hence the $\{|\beta;\mu,\nu\rangle\}$ can be regarded as coherent states of the $\hat{b}$ operator. Later in the term we will show that second-order nonlinear optics can be used to perform Bogoliubov transformations. These systems will allow us to start with a coherent state—such as is emitted by an ideal laser—and transform it into an MUP state with unequal variances in its two quadratures.

Our next task is to study the quadrature-measurement statistics for an MUP state with unequal quadrature-measurement variances. We will do so by building on the Bogoliubov transformation work, which was begun above, generalized to allow for complex-valued $\{\mu,\nu\}$. We define $\hat{b}$ by

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger, \quad \text{where } \mu,\nu \in \mathcal{C} \text{ and } |\mu|^2 - |\nu|^2 = 1. \tag{46}$$

You should verify the $[\hat{b},\hat{b}^\dagger] = 1$ still holds. The result of this commutator-bracket preservation is that $\hat{N}_b \equiv \hat{b}^\dagger\hat{b}$ still has a complete orthonormal set of eigenkets, $\{|n:\mu,\nu\rangle\}$, associated with its non-negative integer eigenvalues, and that

$$\hat{b}|\beta;\mu,\nu\rangle = \beta|\beta;\mu,\nu\rangle, \quad \text{for } \beta,\mu,\nu \in \mathcal{C} \text{ and } |\mu|^2 - |\nu|^2 = 1, \tag{47}$$

still defines the coherent states of $\hat{b}$. To get at the quadrature-measurement statistics—in this case just the mean and variance behavior—we start with the inverse of the Bogoliubov transformation that defines $\hat{b}$, i.e.,

$$\hat{a} = \mu^* \hat{b} - \nu \hat{b}^\dagger. \tag{48}$$

From this we find that

$$\langle \beta; \mu, \nu | \hat{a}(t) | \beta; \mu, \nu \rangle = (\mu^* \beta - \nu \beta^*) e^{-j\omega t}, \tag{49}$$

which is a slight generalization of what we had already shown for the case of μ and ν real valued. The mean values of $\hat{a}_1(t)$ and $\hat{a}_2(t)$ measurements, when the oscillator is in the state $|\beta; \mu, \nu\rangle$, are then just the real and imaginary parts, respectively, of this result. To find the variances of the quadrature measurements, we must work harder. The details for $\langle \Delta \hat{a}_1^2(t) \rangle$ are given below, along with the answer for $\langle \Delta \hat{a}_2^2(t) \rangle$. The derivation of the latter formula is left for you to work out.

We know that $\langle \Delta \hat{a}_1^2(t) \rangle = \langle \hat{a}_1^2(t) \rangle - \langle \hat{a}_1(t) \rangle^2$, with $\langle \hat{a}_1(t) \rangle = \mathrm{Re}[(\mu^* \beta - \nu \beta^*) e^{-j\omega t}]$. For the mean-square term, we have that

$$\langle \hat{a}_1^2(t) \rangle = \frac{\langle \hat{a}^2 \rangle e^{-2j\omega t} + \langle \hat{a}^{\dagger 2} \rangle e^{2j\omega t} + 2\langle \hat{a}^\dagger \hat{a} \rangle + 1}{4}, \tag{50}$$

where we have used $[\hat{a}, \hat{a}^\dagger] = 1$. The average photon number in the state $|\beta; \mu, \nu\rangle$, which appears in the preceding expression, is interesting in its own right:

$$\begin{aligned}
\langle \hat{N} \rangle &= \langle \hat{a}^\dagger \hat{a} \rangle = \langle (\mu^* \hat{b} - \nu \hat{b}^\dagger)^\dagger (\mu^* \hat{b} - \nu \hat{b}^\dagger) \rangle \langle (\mu \hat{b}^\dagger - \nu^* \hat{b})(\mu^* \hat{b} - \nu \hat{b}^\dagger) \rangle & (51) \\
&= |\mu|^2 \langle \hat{b}^\dagger \hat{b} \rangle + |\nu|^2 (\langle \hat{b}^\dagger \hat{b} \rangle + 1) - \mu\nu \langle \hat{b}^{\dagger 2} \rangle - \mu^* \nu^* \langle \hat{b}^2 \rangle & (52) \\
&= (|\mu|^2 + |\nu|^2)|\beta|^2 - 2\mathrm{Re}(\mu^* \nu^* \beta^2) + |\nu|^2 & (53) \\
&= |\mu^* \beta - \nu \beta^*|^2 + |\nu|^2 = |\langle \hat{a} \rangle|^2 + |\nu|^2. & (54)
\end{aligned}$$

The last equality is especially interesting. It shows that the state $|0; \mu, \nu\rangle$ with $\nu \neq 0$,[5] which has $\langle \hat{a} \rangle = 0$, has a non-zero average photon number equal to $|\nu|^2$. To complete our derivation of $\langle \Delta \hat{a}_1^2(t) \rangle$, we note that

$$\langle \hat{a}^2 \rangle = \langle \hat{a}^{\dagger 2} \rangle^* = \langle (\mu^* \hat{b} - \nu \hat{b}^\dagger)^2 \rangle = \mu^{*2} \beta^2 + \nu^2 \beta^{*2} - 2\mu^* \nu |\beta|^2 - \mu^* \nu. \tag{55}$$

Putting everything together, and doing yet more algebra (which will be omitted), yields the final result

$$\langle \Delta \hat{a}_1^2(t) \rangle = \frac{|\mu - \nu e^{-2j\omega t}|^2}{4}. \tag{56}$$

[5]When $\mu = 1$ and $\nu = 0$ we have $\hat{b} = \hat{a}$, so that $|\beta; 1, 0\rangle$ is the coherent state $|\beta\rangle$ of the annihilation operator $\hat{a}$.

A similar derivation for the other quadrature results in

$$\langle \Delta \hat{a}_2^2(t) \rangle = \frac{|\mu + \nu e^{-2j\omega t}|^2}{4}. \tag{57}$$

A natural question to ask, at this point, is when (for what values of t) does $|\beta; \mu, \nu\rangle$ with $|\nu| > 0$ yield a minimum uncertainty product for the quadrature measurements $\hat{a}_1(t)$ and $\hat{a}_2(t)$. This happens when $\mu^* \nu e^{-2j\omega t}$ is positive, in which case

$$\langle \Delta \hat{a}_1^2(t) \rangle = \frac{(|\mu| - |\nu|)^2}{4} < 1/4 \quad \text{and} \quad \langle \Delta \hat{a}_2^2(t) \rangle = \frac{(|\mu| + |\nu|)^2}{4} > 1/4. \tag{58}$$

It also happens when $\mu^* \nu e^{-2j\omega t}$ is negative, in which case[6]

$$\langle \Delta \hat{a}_1^2(t) \rangle = \frac{(|\mu| + |\nu|)^2}{4} > 1/4 \quad \text{and} \quad \langle \Delta \hat{a}_2^2(t) \rangle = \frac{(|\mu| - |\nu|)^2}{4} < 1/4. \tag{59}$$

At other times, however, $|\beta; \mu, \nu\rangle$ is *not* an MUP state for the quadrature operators.

We can get an intuitive feeling for why the Bogoliubov transformation from $\hat{a}$ to $\hat{b}$ creates unequal quadrature variances by looking at the simple case of μ and ν real valued. We then have that

$$\Delta \hat{a}_1 = (\mu - \nu)\Delta \hat{b}_1 \quad \text{and} \quad \Delta \hat{a}_2 = (\mu + \nu)\Delta \hat{b}_2. \tag{60}$$

We know that $|\beta; \mu, \nu\rangle$ is a coherent state of $\hat{b}$, and so it has

$$\langle \Delta \hat{b}_1^2 \rangle = \langle \Delta \hat{b}_2^2 \rangle = 1/4. \tag{61}$$

It thus follows that

$$\langle \Delta \hat{a}_1^2 \rangle = (\mu - \nu)^2/4 \quad \text{and} \quad \langle \Delta \hat{a}_2^2 \rangle = (\mu + \nu)^2/4, \tag{62}$$

which shows that when $\mu\nu > 0$ holds we are *attenuating* the noise in the first quadrature and amplifying the noise in the second quadrature, and vice versa when $\mu\nu < 0$. We say that this noise transformation is *phase sensitive*, because a $\pi/2$-rad phase shift is what distinguishes the two quadratures. We say that this noise transformation is a *squeezing* transformation, because it is reducing the noise (squeezing the noise) in one quadrature and, in order to preserve the minimum uncertainty product, appropriately increasing the noise in the other quadrature. Thus the state $|\beta; \mu, \nu\rangle$ is referred to as a *squeezed state*. Also, because the zero-mean coherent state $|0\rangle$ is the vacuum state, we call the zero-mean squeezed state, $|0; \mu, \nu\rangle$, a *squeezed vacuum* state. The squeezed vacuum state plays an essential role in the quantum waveguide tap that was described in Lecture 1.

We'll conclude today's lecture with a qualitative look at two special cases of squeezed states, shown on Slide 12. In one case, we have reduced the noise that is in phase with the mean, obtaining an *amplitude-squeezed* state. In the other case, we have reduced the noise that is in quadrature with ($\pi/2$ rad out of phase from) the mean, obtaining a phase-squeezed state.

[6] Note that $\langle \Delta \hat{a}_1^2(t) \rangle \langle \Delta \hat{a}_2^2(t) \rangle = (|\mu| - |\nu|)^2 (|\mu| + |\nu|)^2/16 = (|\mu|^2 - |\nu|^2)^2/16 = 1/16$, as advertised.

The Road Ahead

The squeezed states are an important class of non-classical states, i.e., putting a single-mode electromagnetic field into one of these states results in quantum photodetection statistics that *cannot* be described by classical electromagnetism and shot noise. We are almost, but not quite, ready to begin a treatment of quantum photodetection for single-mode fields. The last things to learn before doing so will be covered in the next lecture, where we will introduce quantum characteristic functions, and describe what it means to "measure" the annihilation operator.[7]

[7] Because $\hat{a}$ is a non-Hermitian operator whose real and imaginary parts are non-commuting observables, nothing we have done so far implies what it means for $\hat{a}$ can be "measured."