Lecture 6. Stabilizers II

1) Stabilizers (Review)
2) The Normalizer
3) Gottesman – Knill
4) Computing on codes
5) Teleportation Stabilizers

I) Stabilizers

- G_n Pauli Group (n qubits)
  if $g, h \in G$ then
  either $[g, h] = 0$ or $\{g, h\} = 0$

- A stabilizer for $\{|\psi_\ell\rangle\} = V_S$ is the set S:
  vectors, vector space

$$S = \{ g \in G \mid g|\psi\rangle = |\psi\rangle, \ \forall |\psi\rangle \in V_S \}$$

- By convention $-I \notin S$
- Stabilizers are Abelian.

0 a) $V_S = \{|00\rangle\}$

$$S = \{ZI, II, IZ, ZZ\}$$
$$= \langle IZ, ZI \rangle$$

b) $V_S = \left\{ \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right\}$ ← $\dim V_S = 2^k$

$$S = \{XX, ZZ, YY\}$$
$$= \langle XX, ZZ \rangle$$

\# qubits $= n$

\# min generators $= n - k$

c) $V_S = \Phi$ (null)

$$\cancel{S = \{X, Z\}}$$

d) $V_S = \{ |000\rangle, |111\rangle \}$

$$S = \langle ZZI, IZZ \rangle$$

e) $V_S = \{ (|000\rangle + |111\rangle)^{\otimes 3}, (|000\rangle - |111\rangle)^{\otimes 3} \}$

$$S = \langle ZZI\, I^{\otimes 6}, IZZI^{\otimes 6}, \cdots \\ X^{\otimes 6} I^{\otimes 3}, I^{\otimes 3} X^{\otimes 6} \rangle$$

f) $S = \langle XX \rangle$

$$V_S = \left\{ \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \right\}$$

The Normalizer

$|\psi\rangle \xrightarrow{U} U|\psi\rangle$ $\qquad\qquad$ $S \xrightarrow{U} USU^{\dagger}$

Proof : $U|\psi\rangle \rightarrow (USU^{\dagger})U|\psi\rangle$
$= US|\psi\rangle$
$= U|\psi\rangle$

- The Normalizer of S
  $N(S) = \{ g \in G \mid ghg^{\dagger} \in S, \ \forall h \in S \}$

- Lemma : $N(S) = \{ g \in G \mid [g, h] = 0, \ \forall h \in S \}$

Proof : Recall $gh = \pm hg$

$$ghg^\dagger = \pm g g^\dagger h = \pm h = +h$$

but $-I \notin S$, thus

$$\Longrightarrow [g, h] = 0$$

a) Stabilizer $S = \langle IZ, ZI \rangle$

$N(S) = \{ II, ZZ, ZI, IZ \}$

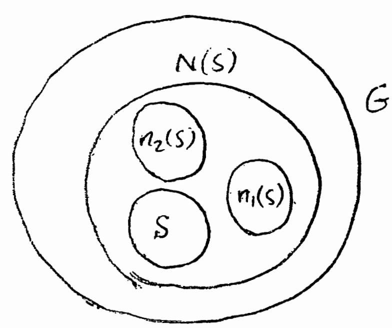
$$\boxed{S \subseteq N(S)}$$

b) $S = \langle XX \rangle$

$N(S) = \{ XI, IX, ZZ, YY \}$

c) $S = \langle IXX, IZZ \rangle$ ⟵ $\dim(V_S) = 1$ qubit

$N(S) = \{ \underbrace{XII}_{\bar{X}}, \underbrace{ZII}_{\bar{Z}}, YII \}$



$G_1 = \{X, Y, Z\}$

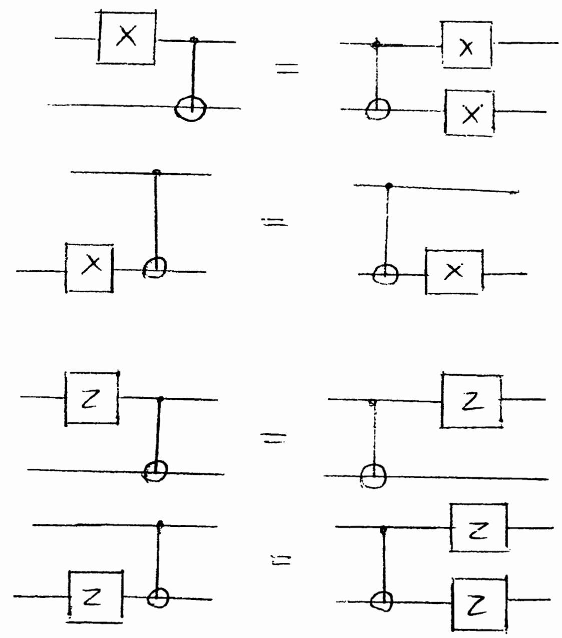
I	X	Z	H	S (phase gate)
X	X	−X	Z	Y
Y	−Y	−Y	−Y	X
Z	−Z	Z	X	Z

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \sim \sqrt{Z}$$

- The Clifford Group

  --> What is the normalizer of the Pauli Group?

  CNOT Gate



- The Clifford group $C_2 \equiv N(G) = \langle H, S, CNOT \rangle$

- Gottesman–Knill

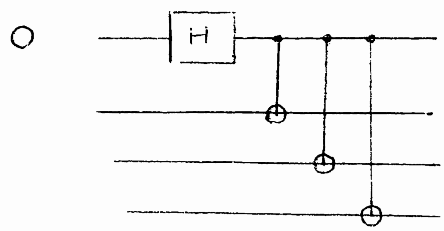
  1) Suppose $UgU^\dagger \in G_n$, $\forall g \in G_n$

     ↳ unitary

     Then U can be constructed $O(n^2)$ CNOT, H, S

  2) Any Q. Circuit comprised of CNOT, H, S and uses $|0\rangle^{\otimes n}$ and meas. in the comp. basis, and any amount of ~~clas~~ classical feedback

     can be efficiently classically simulated!



$\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$

Computing on Codes

$\Rightarrow$ 5 qubit code

$$S = \begin{cases} XZZXI \\ ZZXIX \\ ZXIXZ \\ XIXZZ \end{cases}$$

$N(S) = X^{\otimes 5}, Z^{\otimes 5}$

$H^{\otimes 5} \notin N(S)$

$\Rightarrow$ 7 qubit Steane code

$$S = \begin{cases} IIIZZZZ \\ IZZIIZZ \\ ZIZIZIZ \\ \quad \parallel \\ Z \rightarrow X \end{cases}$$

$N(S) = X^{\otimes 7}, Z^{\otimes 7}, H^{\otimes 7}, CNOT$

$\Rightarrow$ 9 qubit Shor code $[[9,1,3]]$

$$S = \langle X^{\otimes 6} I^{\otimes 3}, I^{\otimes 3} X^{\otimes 6}, Z^{\otimes 2} I^{\otimes 7}, IZ^{\otimes 2} I^{\otimes 6}, \cdots \rangle$$

$g_1 = XXXXXXIII$

$g_2 = IIIXXXXXX$

$g_3 = ZZII \cdots$

$g_4 = IZZI \cdots$

$g_5 = III ZZII \cdots$

$g_6 = IIIIZZI \cdots$

$g_7 = IIIIII ZZI$

$g_8 = I \cdots IIZZ$

$N(S) =$

$\bar{Z} = X^{\otimes 9}$

$\bar{X} = Z^{\otimes 9}$

- Bacon-Shor codes

$[[9,5,2]]$

$$S = \begin{bmatrix} g_4' = g_4 g_6 g_8 = IZZIZZIZZ \\ g_3' = g_3 g_5 g_7 = ZZIZZIZZI \\ g_1 = \\ g_2 = \end{bmatrix}$$

$$N(S) = \left.\begin{matrix} g_3 = \bar{Z}_1 & \bar{X}_1 = XII\ III\ XII \\ g_4 = \bar{Z}_2 & \bar{X}_2 = IIX\ III\ IIX \\ g_5 = \bar{Z}_3 & \vdots \\ g_6 = \bar{Z}_4 & \\ Z^{\otimes 9} = \bar{Z}_5 & \bar{X}_5 = X^{\otimes 9} \end{matrix}\right] \rightarrow \text{Discard "Gauge Qubits"}$$

"Subsystem Code"

$[[9,1,3]]$

▷ Another picture:

1 2 3
4 5 6
7 8 9

$g_1 = \begin{matrix} IZZ \\ IZZ \\ IZZ \end{matrix}$ $\quad X_1 = \begin{matrix} XII \\ III \\ XII \end{matrix}$

$X_4 = \begin{matrix} IIX \\ III \\ IIX \end{matrix}$

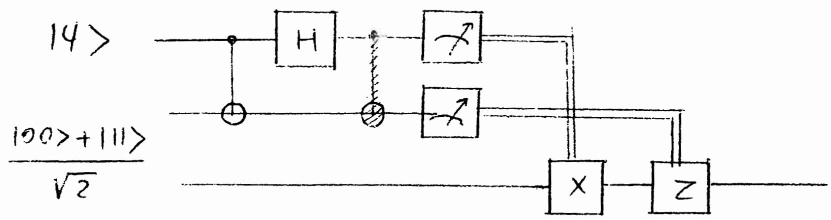
$\bar{Z}$

1 2 3
4 5 6
7 8 9

$\bar{X}$

Teleportation in the Stabilizer formalism

Recall:



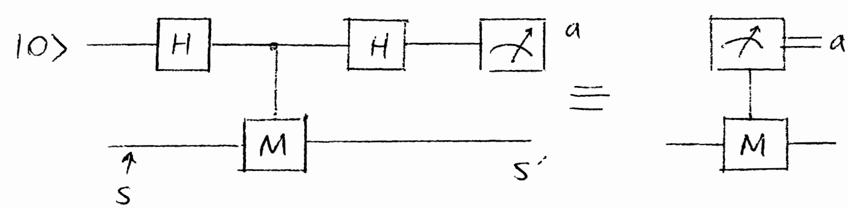
Measurements

$\Rightarrow$ Suppose $S = \langle g_1, \cdots, g_n \rangle$ and we measure M $(M^2 = I)$

without loss of generality assume

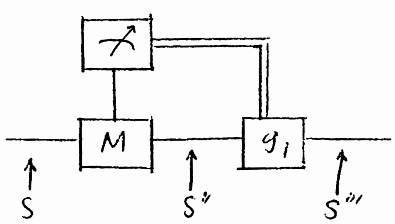
$$\{M, g_1\} = 0, \qquad [M, g_k] = 0 \quad \forall k > 1$$

Operator measurement:



$$S \rightarrow S' = \langle (-1)^a M, g_2, \cdots, g_n \rangle$$

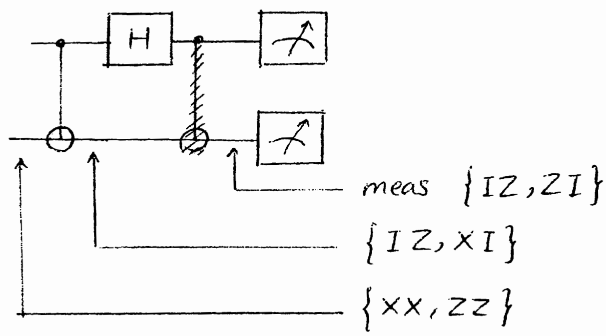
▶ Projection into the $+1$ Eigenspace:



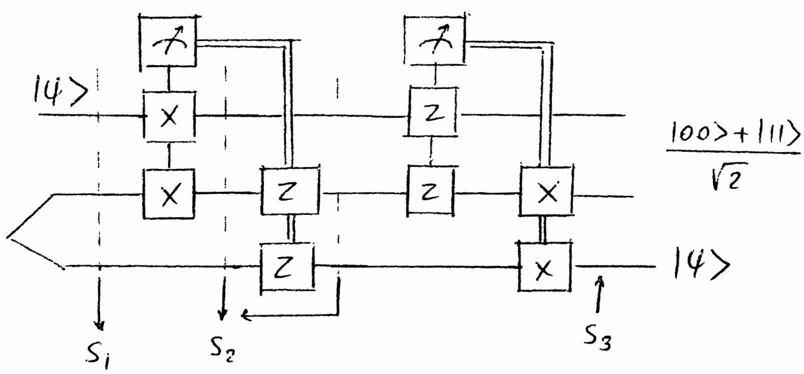
$S' \rightarrow S'' = \langle M, g_2 \cdots g_n \rangle$

Note: $g_1^\dagger (-M) g_1 = g_1^\dagger g_1 M = M$

▸ *Bell basis measurement*



IN	CNOT
IZ	ZZ
XI	XX



$S_1 = \langle IXX, IZZ \rangle \qquad \bar{X}_1 = XII \ , \bar{Z}_1 = ZII$

$S_2 = \langle IXX , XXI \rangle \quad \bar{X}_2 = XII \ , \bar{Z}_2 = \bar{Z}_1 \cdot g_1 \quad \longleftarrow$ measure XXI

$\Rightarrow \bar{Z}_2 = ZZZ \qquad$ Fix using $IZZ = g_1$

$S_3 = \langle ZZI, XXI \rangle \quad \bar{X}_3 = \bar{X}_2 \cdot IXX = XXX \quad \longleftarrow$ measure ZZI

$\bar{Z}_3 = \bar{Z}_2 = ZZZ \qquad$ Fix IXX

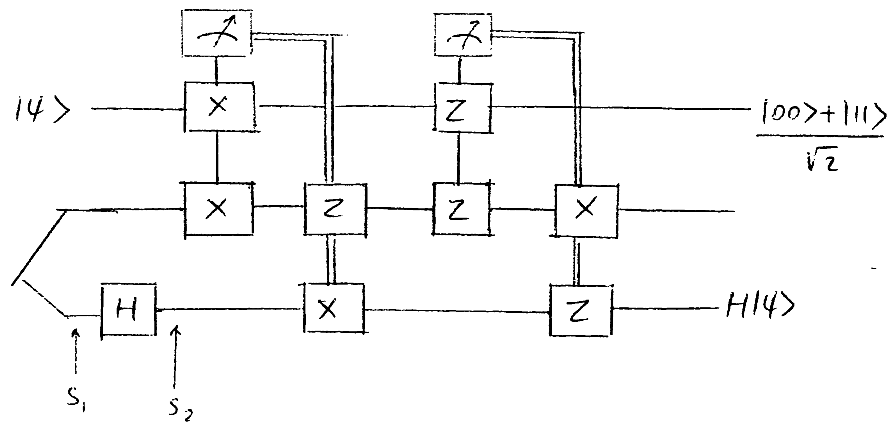
$\bar{X}_3 \doteq IIX$

$\bar{Z}_3 \doteq IIZ$

<u>Teleporting an H</u>

$S_1 = \langle IXX , IZZ \rangle \qquad \bar{X}_1 = XII \ , \ \bar{Z}_1 = ZII$

$\xrightarrow{H} \quad S_2 = \langle IXZ , IZX \rangle \ , \bar{X}_2 = XII \ , \ \bar{Z}_2 = ZII$



measure XXI $\qquad S_3 = \langle IXZ, XXI \rangle$

Fix IZX $\qquad \bar{X}_3 = XII$

$\bar{Z}_3 = ZZX$

measure ZZI $\qquad S_4 = \langle ZZI, XXI \rangle$

Fix IXZ $\qquad \bar{X}_4 = XXZ \doteq IIZ$

$\bar{Z}_4 = ZZX \doteq IIX$

$\circ\ S_1 = \langle IZ \rangle \quad \bar{X}_1 = XI \quad \bar{Z}_1 = ZI \xrightarrow{CNOT} S_2 = \langle ZZ \rangle \quad \bar{X}_2 = XX \quad \bar{Z}_2 = ZI \xrightarrow[\text{Fix } ZZ]{\text{measure } IY} S_3 = \langle \cancel{X} IY \rangle$

$\bar{X}_3 = -YY \doteq -YI$

$\bar{Z}_3 = ZI$

S - gate