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Quantum Information Science II : 4/13/2006

Lecture 17: Entanglement as a Physical Resource

(1) The resource model of QIT.
(2) Entanglement : def / meas.
(3) Fungibility: Compression / Dilution
(4) QECC
(5) Topics : - Ent. capacity of gates
- mixed state entanglement

I.

Resources: - noisy classical channel $I(x;y)$
- shared randomness $H(x)$
- noisy quantum channel Φ
- entanglement.

Uses of Ent: - Teleportation: 1 ebit + 2cbits → 1 qbit
- SDC: 1 ebit + 2qbits → 2cbits
- clock synchronization
- distributed computation
- cryptography.

Is entanglement a resource?

Ex $\sqrt{0.9}\,|00\rangle + \sqrt{0.1}\,|11\rangle$ different from $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$?

Resource: A and B are equivalent if $A \to B$ and $B \to A$ is possible.

Thm $|\psi\rangle$ and $|\phi\rangle$ are equiv. under LOCC iff ψ majorizes ϕ and ϕ majorizes ψ: i.e. if eigenvalues of the reduced density matrices are same.

Asymptotic equivalence: pounds → dollars + fixed charge

def A and B are asymptotically equivalent if $\exists$ a ratio R s.t. $\forall \epsilon, \delta > 0, \exists N, \forall n > N$:

$$A^{n(R+\delta)} \longrightarrow B^n$$
$$B^n \longrightarrow A^{n(R-\delta)}$$

with error $< \epsilon$.

II. Entanglement and Measures

def A bipartite state $|\psi_{AB}\rangle$ of a composite system is entangled iff $\nexists\ |\psi_A\rangle, |\psi_B\rangle$ s.t. $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

Measures

$\Rightarrow$ Entropy. Let $\rho_A = \mathrm{Tr}_B(|\psi_{AB}\rangle\langle\psi_{AB}|)$

def $E(|\psi_{AB}\rangle) \equiv S(\rho_A) = S(\rho_B)$ "The Entanglement".

Ex $\psi_{AB} = 00 + 11$

$$\rho_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \qquad S(\rho_A) = 1 \text{ "ebit"}$$

Ex

$$(00+11) \otimes (00+11) = \overbrace{00}^{A}\underbrace{00}_{B} + 0011 + 1100 + 1111$$
$$= \overset{A}{0}\overset{B}{0}00 + 0101 + 1010 + 1111$$
$$= \sum_{x=0}^{3} |xx\rangle$$

III. Fungibility

Claim: All bipartite entangled pure states are asymp. equiv.

Proof ($\Phi \equiv |00\rangle + |11\rangle$: "gold standard")

Part 1 Ent. concentration : $\underset{\substack{\text{arb.} \\ \text{state}}}{\psi}{}^n \rightarrow \Phi^{n(E(\psi)-\delta)}$

(a.k.a. purification)

Part 2 Entanglement Dilution

$$\Phi^{n(E(\psi)+\delta)} \rightarrow \psi^n$$

$\forall \epsilon, \delta > 0$, $\exists N$ s.t. $\forall n > N$ ......

Concentration

Suppose $|\psi\rangle = \sqrt{1-p}\,|00\rangle + \sqrt{p}\,|11\rangle$

Recall $E(\psi) = S(\mathrm{Tr}_A(\psi))$

$$= -p\log p - (1-p)\log(1-p)$$

$$= H_2(p)$$

$$\underbrace{\psi^n}_{\text{arb.}} \longrightarrow \underbrace{\Phi}_{\text{EPR}}^{\,n(E(\psi)-\delta)}$$

$$\psi^n = \sum_{x\in\{0,1\}^n} (1-p)^{\frac{n-|x|}{2}}\, p^{\frac{|x|}{2}}\, |xx\rangle \quad (|x| = \#\text{ of ones})$$

$$= \sum_{w=0}^{n} (1-p)^{\frac{n-w}{2}}\, p^{w} \sum_{|x|=w} |xx\rangle$$

$$= \sum_{w} \sqrt{\binom{n}{w}(1-p)^{n-w}p^w}\, |S_w\rangle$$

$$S_w = \sqrt{\frac{1}{\binom{n}{w}}} \sum_{|x|=w} |xx\rangle$$

Each $|S_w\rangle$ is $\sim \log\binom{n}{w}$ EPR pairs.

Procedure: A and B both measure w
$\Rightarrow$ collapses onto $|S_w\rangle$

$\mathrm{Prob}(w) = \binom{n}{w}(1-p)^{n-w}p^w \simeq$ Gaussian mean: np
variance: $np(1-p)$

$$\log\binom{n}{w} \sim \log\binom{n}{np} \simeq nH_2(p) = nE - o(\log\sqrt{n})$$

(using $\log N! \simeq N\log N - N$)

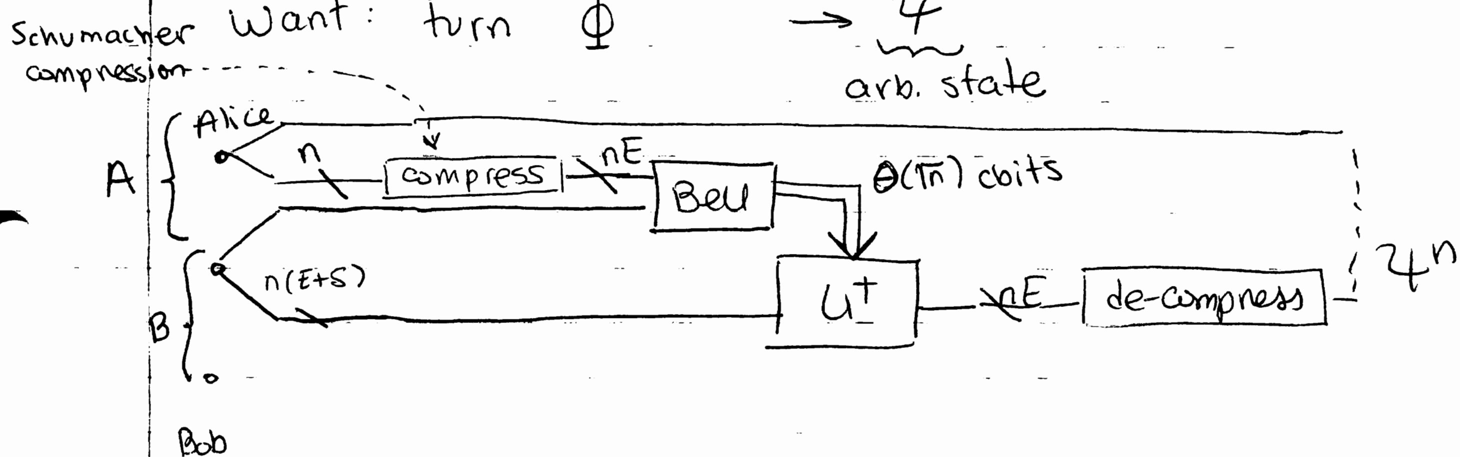
choose $n\delta = \omega(\sqrt{n})$

<u>Def</u> $D(|\psi\rangle) = \lim_{n\to\infty} \dfrac{\text{best \# of EPR pairs distillable from } \psi^n}{n}$

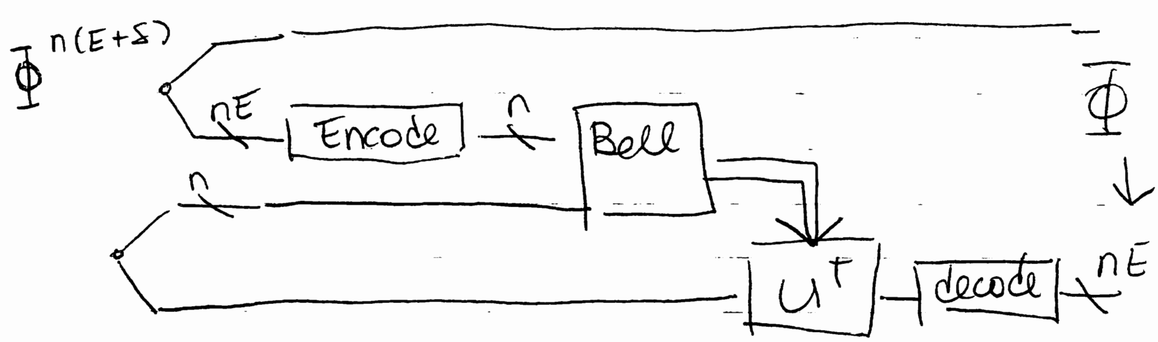
"The Distillable entanglement"
$= E(\psi)$ [this is false for mixed states]

<u>Dilution</u>

Want: turn $\Phi^{n(\delta+E)} \rightarrow \underbrace{\psi^n}_{\text{arb. state}}$

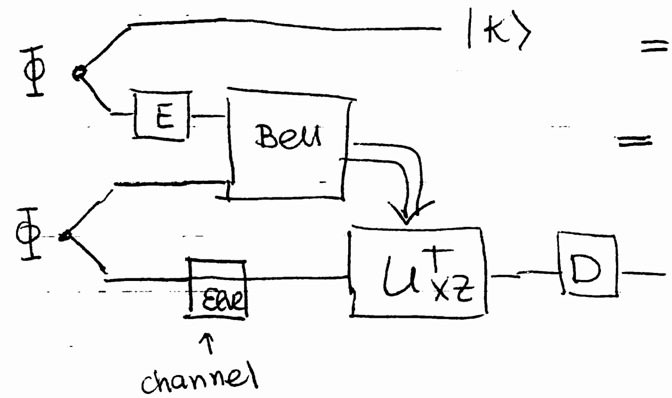


IV. Relationship to QECC
⇒ Dilution used teleportation: noiseless channel
⇒ Concentration using telep. noisy channel.



Required code parameters $[[n, nE, d]]$?

⇒ Case: CSS codes

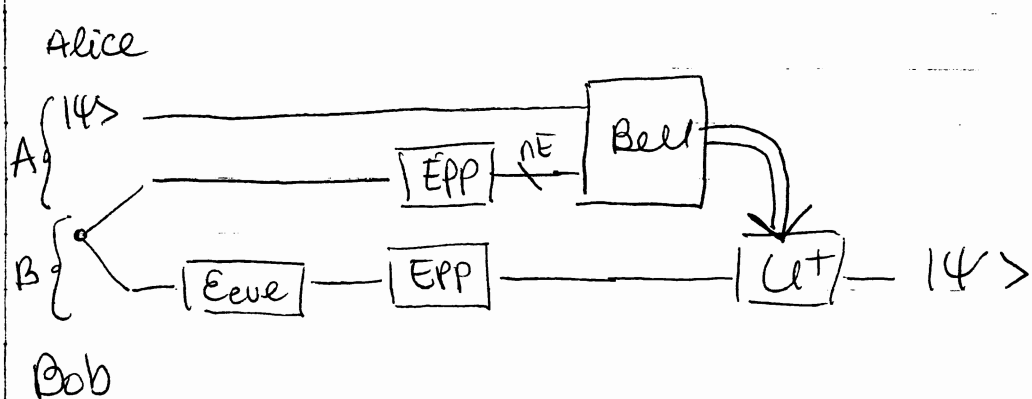


$= |k\rangle - \boxed{E_{xz}} - \boxed{\mathcal{E}} - \boxed{D_{xz}} - \rho$

$= |k\rangle - \underbrace{\boxed{E} - \boxed{U_{xz}}}_{\text{random CSS code}} - \boxed{\mathcal{E}} - \underbrace{\boxed{U^\dagger_{xz}} - \boxed{D}}_{\text{random decoder}} - \rho$

random CSS code ↔ random decoder

Channel coding using entanglement purification



versus $|\psi\rangle - \boxed{E} - \boxed{\mathcal{E}_{eve}} - \boxed{D} - \tilde{\psi}\rangle$

= use of EPP can work when coding fails!

Ex: depolarizing channel

$$\mathcal{E}(\rho) = p\rho + \frac{1-p}{3}(X\rho X + Z\rho Z + Y\rho Y)$$

Fact: if $p < 3/4$ then capacity of $\mathcal{E}$ is zero.
(pf. quantum singleton bound)

EPP: get $E(\psi)$ EPR pairs

$\exists \rho_0$ s.t. $E(\mathcal{E}(\rho_0)) > 0$ when $p \leq 3/4$.

EPP $\cong$ two-way classical communication!

V Topics

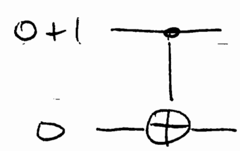
=> Gates as a resource

What is the entangling capacity of a unitary gate?

<u>state</u>

$\rho = I/2^n + \epsilon\,\Phi$ (for ϵ suff small $\rho \to$ unentangled)

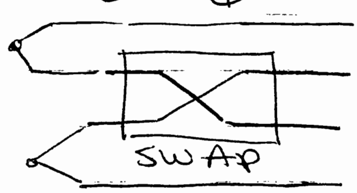
<u>Ex</u>



00+11

<u>CNOT</u>

1 ebit



<u>SWAP</u>

2 ebit

<u>Def</u> $E(U) = \lim_{n\to\infty} \left(\dfrac{\text{max \# ent epp created } n \text{ uses of } U}{n} \right)$

<u>claim</u>

$$E(U) = \sup_{|\psi\rangle} E(U|\psi\rangle) - E(|\psi\rangle)$$

for $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$

=> mixed states <u>def</u> ρ_{AB} separable iff $\exists \{\rho_k^A, \rho_k^B\}$

s.t. $\rho_{AB} = \sum_k p_k\, \rho_k^A \otimes \rho_k^B$

p_k prob.

Separable <=> non-entangled

$\Leftrightarrow$ ρ_{AB} separable iff $\forall$ positive maps $\mathcal{E}: H_B \to H_B$, $(I\otimes\mathcal{E})\rho \geq 0$

ρ entangled $\exists$ map $\mathcal{E}$ s.t. $(\psi\otimes\mathcal{E})\rho < 0$

Ex $\mathcal{E}_{PT}: \rho \to \rho^T$

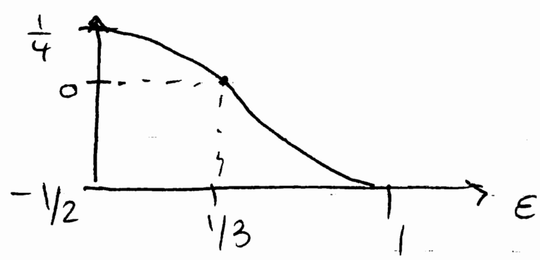
"positive partial transpose test" PPT

Claim

If ρ is separable $\Rightarrow (I \otimes \mathcal{E}_{PT})(\rho) \geq 0$

Ex $\rho = (1-\epsilon)\frac{I}{4} + \epsilon \, \overline{\Phi}$

$\min(\mathrm{eig}[(I \otimes \mathcal{E}_{PT})(\rho)])$



Ex $\rho = (1-\epsilon)\frac{I}{2^n} + \epsilon \frac{|0^n\rangle + |1^n\rangle}{\sqrt{2n}}$

$\epsilon < \frac{1}{1+2^{n-1}}$ $\qquad$ $\epsilon > \frac{1}{1+2^{n/2}}$

separable $\qquad$ ↑ $\qquad$ entangled

gap?

$E_f(\rho) \geq E(\rho) \geq D(\rho)$

entanglement of formation: # EPR pairs to create