LECTURE 13

- Readings: Section 4.3, 4.4

Lecture outline

- Conditional Expectation:
  - Law of iterated expectations
  - Law of conditional variances

Do you want a job at Microsoft?
The deadly hat problem

100 people are lined up in a straight line. Either a blue or red hat is placed randomly on each person's head. An executioner will walk up from the back of the line and ask each person about the color of her hat. She is supposed to guess (correctly) the color of her own hat, or else she will die! Each person can only see the hats in front of her, and she can hear the answer of the person behind her.

The 100 people are allowed to get the advise of a job applicant. The applicant that saves the biggest number of lives gets the job. How many can you save?

Answer

- Person 100 says:
  - Blue if No. of Blue is odd
  - Red if No. of Blue is even
- Next person: counts the number of blue in front and decides if she is blue or red.
- Third person: does the same!

Conditional Expectation

- ... of a r.v. X given the value y of a r.v. Y :

$$\boxed{\mathbf{E}[X|Y=y] = \begin{cases} \sum_x x \; p_{X|Y}(x|y) \\ \int_{-\infty}^{\infty} x \; f_{X|Y}(x|y)dx \end{cases}}$$

- $\mathbf{E}[X|Y]$ seen as a function of Y , is a r.v.!

- Recall: **Stick-Breaking Example**
  - Break a stick of length ℓ uniformly at a point X .
  - Break the remaining stick uniformly at a point Y .

$$\mathbf{E}[Y|X=x] = \frac{x}{2} \qquad \text{(number)}$$

$$\mathbf{E}[Y|X] = \frac{X}{2} \qquad \text{(random variable)}$$

Law of Iterated Expectations

$$\boxed{\mathbf{E}\Big[\mathbf{E}[X|Y]\Big] = \left\{ \begin{array}{c} \sum_y \mathbf{E}[X|Y=y]\ p_Y(y) \\ \int_{-\infty}^{\infty} \mathbf{E}[X|Y=y]\ f_Y(y)dy \end{array} \right\} = \mathbf{E}[X]}$$

- **Stick-Breaking Example:**

$$\mathbf{E}[Y|X] = \frac{X}{2}$$

$$\mathbf{E}[Y] = \mathbf{E}\Big[\mathbf{E}[Y|X]\Big] = \mathbf{E}\left[\frac{X}{2}\right] = \frac{\ell}{4}$$

  - Contrast this to computing the PDF of Y first, which is what we did in Lecture 9.

Conditional Variance

- … of a r.v. X given the value y of a r.v. Y :

$$\boxed{\mathsf{Var}(X|Y=y) = \mathbf{E}\left[(X - \mathbf{E}[X|Y=y])^2\,|Y=y\right]}$$

- $\mathsf{Var}(X|Y)$ seen as a function of Y , is a r.v. too.

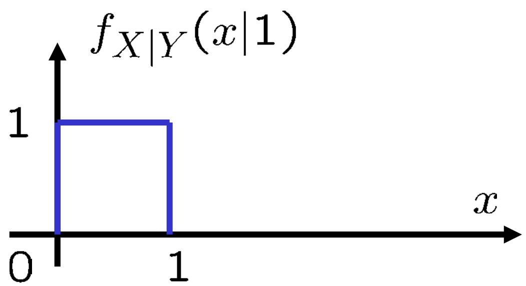
- Interesting formula:

$$\mathsf{Var}(X) = \mathbf{E}[\mathsf{Var}(X|Y)] + \mathsf{Var}(\mathbf{E}[X|Y])$$

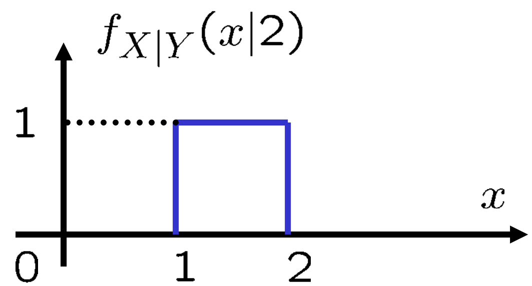
Example (1)

- Throw a biased coin, with $\mathbf{P}(\mathsf{H}) = \dfrac{1}{3}$:
  - If H: choose a number uniformly in [0, 1].
  - If T: choose a number uniformly in [1, 2].

- Using random variables:



$p_Y(1) = \mathbf{P}(Y = 1) = \dfrac{1}{3}$



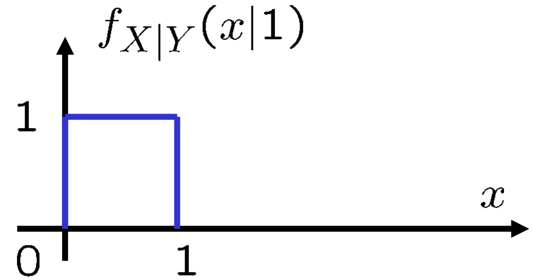
$p_Y(2) = \mathbf{P}(Y = 2) = \dfrac{2}{3}$

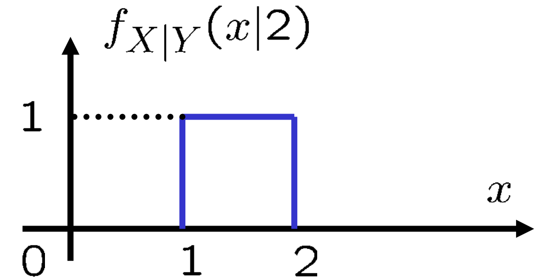
- We're interested in $\mathsf{Var}(X)$.

- Note that this example is part continuous (X), and part discrete (Y). Our results hold anyway.

Example (2)

$$p_Y(1) = \frac{1}{3} \qquad p_Y(2) = \frac{2}{3}$$





- Use: $\mathrm{Var}(X) = \mathbf{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbf{E}[X|Y])$

$$\left.\begin{aligned} \mathrm{Var}(X|Y=1) &= 1/12 \\ \mathrm{Var}(X|Y=2) &= 1/12 \end{aligned}\right\} \Rightarrow \ \mathbf{E}[\mathrm{Var}(X|Y)] = 1/12$$

$$\left.\begin{aligned} \mathbf{E}[X|Y=1] &= \ 1/2 \\ \mathbf{E}[X|Y=2] &= \ 3/2 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[X] = \ 7/6 \\ &\mathbf{E}[\mathbf{E}[X|Y]^2] = 19/12 \end{aligned}\right.$$

$$\mathrm{Var}(\mathbf{E}[X|Y]) = \ \mathbf{E}[\mathbf{E}[X|Y]^2] - \mathbf{E}[\mathbf{E}[X|Y]]^2 = \ 2/9$$

- Finally: $\mathrm{Var}(X) = \ 1/12 + 2/9 = 11/36$

The Challenge Problem: BBall Party

Your Guests are all BBall fans and they wear BBall Caps. There is a total of s teams in the league. Everyone of your guests is equally likely to be a fan of any one of these teams.

Compute the expected number of people who will pick a cap from their own team!

- C =The number of people picking up a cap from their own tean

- C_i =The number of people of team i that pick a hat from their Own team

$$C = C_1 + \ldots + C_s$$

Conditional Expectation

- $$\mathbf{E}[C] = s\mathbf{E}[C_i]$$

- Let R_i be the number of people of team i

- $\mathbf{E}[C_i | R_i = R] = R\frac{R}{n} = \frac{R^2}{n}$

  (R people with probability $\frac{R}{n}$ of picking the correct Hat)

- $$\mathbf{E}[C_i] = \mathbf{E}\Big[\mathbf{E}[C_i | R_i]\Big] = \mathbf{E}[\frac{R_i^2}{n}]$$

binomial $$= \frac{1}{n}[n\frac{1}{s}(1 - \frac{1}{s}) + (n\frac{1}{s})^2]$$

$$\mathbf{E}[C] = \frac{n-1+s}{s^2}$$