MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Problem Set 6: Solutions
Due: April 5, 2006

1. X is the mixture of two exponential random variables with parameters 1 and 3, which are selected with probability 1/3 and 2/3, respectively. Hence, the PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{3} \cdot e^{-x} + \frac{2}{3} \cdot 3e^{-3x} & \text{for } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. X is a mixture of two exponential random variables, one with parameter λ and one with parameter μ. We select the exponential with parameter λ with probability p, so the transform is $M_X(s) = p\frac{\lambda}{\lambda-s} + (1-p)\frac{\mu}{\mu-s}$. Note that the transform only exists for $s < \min\{\lambda, \mu\}$.

3. (a) The definition of the transform is

$$M_Z(s) = \mathbf{E}[e^{sZ}]$$

Therefore, we know the following must be true:

$$M_Z(0) = \mathbf{E}[e^{0Z}] = \mathbf{E}[1] = 1.$$

So in our case

$$M_Z(0) = \frac{a}{8} = 1$$

and

$$a = 8.$$

(b) We approach this problem by first finding the PDF of Z using partial fraction expansion:

$$M_Z(s) = \frac{8-3s}{s^2-6s+8} = \frac{A}{s-4} + \frac{B}{s-2}$$
$$A = (s-4)M_Z(s)\Big|_{s=4} = \frac{8-3s}{s-2}\Big|_{s=4} = -2$$
$$B = (s-2)M_Z(s)\Big|_{s=2} = \frac{8-3s}{s-4}\Big|_{s=2} = -1.$$

Thus,

$$M_Z(s) = \frac{-2}{s-4} + \frac{-1}{s-2} = \frac{1}{2}\left(\frac{4}{4-s} + \frac{2}{2-s}\right)$$

and

$$f_Z(z) = \begin{cases} \frac{1}{2}\left(4e^{-4z} + 2e^{-2z}\right) & \text{for } z \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

From this we get

$$\mathbf{P}(Z \geq 0.5) = \int_{0.5}^{\infty} \frac{1}{2}(4e^{-4z} + 2e^{-2z})dz = \boxed{\frac{e^{-2}}{2} + \frac{e^{-1}}{2}}.$$

(c) $\mathbf{E}[Z] = \int_0^\infty \frac{z}{2}(4e^{-4z} + 2e^{-2z})dz = \frac{1}{2}(\int_0^\infty 4ze^{-4z}dz + \int_0^\infty 2ze^{-2z}dz) = \frac{1}{2}(\frac{1}{4} + \frac{1}{2}) = \boxed{\frac{3}{8}}$

(d) $\mathbf{E}[Z] = \frac{d}{ds}M_Z(s)\Big|_{s=0} = \frac{d}{ds}(\frac{2}{4-s} + \frac{1}{2-s})\Big|_{s=0} = \frac{2}{(4-s)^2} + \frac{1}{(2-s)^2}\Big|_{s=0} = \boxed{\frac{3}{8}}$

(e) $\text{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2$
$\mathbf{E}[Z^2] = \int_0^\infty \frac{z^2}{2}(4e^{-4z} + 2e^{-2z})dz = \frac{1}{2}(\int_0^\infty 4z^2e^{-4z}dz + \int_0^\infty 2z^2e^{-2z}dz) = \frac{1}{2}(\frac{2}{4^2} + \frac{2}{2^2}) = \frac{5}{16}$
$\text{var}(Z) = \frac{5}{16} - (\frac{3}{8})^2 = \boxed{\frac{11}{64}}$

(f) $\mathbf{E}[Z^2] = \frac{d^2}{ds^2}M_Z(s)\Big|_{s=0} = \frac{d^2}{ds^2}(\frac{2}{4-s} + \frac{1}{2-s})\Big|_{s=0} = \frac{4}{(4-s)^3} + \frac{2}{(2-s)^3}\Big|_{s=0} = \frac{5}{16}$
$\text{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2 = \frac{5}{16} - (\frac{3}{8})^2 = \boxed{\frac{11}{64}}$

4. (a) Since it is impossible to get a run of n heads with fewer than n tosses, it is clear that $p_T(k) = 0$ for $k < n$. In addition, the probability of getting n heads in n tosses is q^n so $p_T(n) = q^n$. Lastly, for $k \geq n+1$, we have $T = k$ if there is no run of n heads in the first $k-n-1$ tosses, followed by a tail, followed by a run of n heads, so

$$p_T(k) = \mathbf{P}(T > k-n-1)(1-q)q^n = \left(\sum_{i=k-n}^{\infty} p_T(i)\right)(1-q)q^n.$$

(b) We use the PMF we obtained in the previous part to compute the moment generating function. Thus,

$$\begin{aligned} M_T(s) &= \mathbf{E}[e^{sT}] = \textstyle\sum_{k=-\infty}^{\infty} p_T(k)e^{sk} \\ &= q^n e^{sn} + (1-q)q^n \textstyle\sum_{k=n+1}^{\infty}\sum_{i=k-n}^{\infty} p_T(i)e^{sk}. \end{aligned}$$

We observe that the set of pairs $\{(i,k) \mid k \geq n+1, i \geq k-n\}$ is equal to the set of pairs $\{(i,k) \mid i \geq 1, n+1 \leq k \leq i+n\}$, so by reversing the order of the summations, we have

$$\begin{aligned} M_T(s) &= q^n e^{sn} + (1-q)q^n \textstyle\sum_{i=1}^{\infty}\sum_{k=n+1}^{i+n} p_T(i)e^{sk} \\ &= q^n e^{sn}\left(1 + (1-q)\textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i} p_T(i)e^{sk}\right) \\ &= q^n e^{sn}\left(1 + (1-q)\textstyle\sum_{i=1}^{\infty} p_T(i)\frac{e^s - e^{s(i+1)}}{1-e^s}\right) \\ &= q^n e^{sn}\left(1 + \frac{(1-q)e^s}{1-e^s}\textstyle\sum_{i=1}^{\infty} p_T(i)(1-e^{si})\right). \end{aligned}$$

Now, since $\sum_{i=1}^{\infty} p_T(i) = 1$ and, by definition, $\sum_{i=1}^{\infty} p_T(i)e^{si} = M_T(s)$, it follows that

$$M_T(s) = q^n e^{sn}\left(1 + \frac{(1-q)e^s}{1-e^s}(1 - M_T(s))\right).$$

Rearrangement yields

$$\begin{aligned} M_T(s) &= \frac{1 + \frac{(1-q)e^s}{1-e^s}}{\frac{1}{q^n e^{sn}} + \frac{(1-q)e^s}{1-e^s}} = \frac{q^n e^{sn}((1-e^s) + (1-q)e^s)}{1 - e^s + (1-q)q^n e^{s(n+1)}} \\ &= \frac{q^n e^{sn}(1-qe^s)}{1-e^s+(1-q)q^n e^{s(n+1)}}. \end{aligned}$$

(c) We have

$$\begin{aligned} \mathbf{E}[T] &= \frac{d}{ds}M_T(s)\Big|_{s=0} \\ &= \Big\{ \frac{[1-e^s+(1-q)q^n e^{s(n+1)}][nq^n e^{sn}(1-qe^s) - qe^s q^n e^{sn}]}{(1-e^s+(1-q)q^n e^{s(n+1)})^2} \end{aligned}$$

$$
\begin{aligned}
& - \left. \frac{q^n e^{sn}(1-qe^s)(-e^s+(n+1)(1-q)q^n e^{s(n+1)}]}{(1-e^s+(1-q)q^n e^{s(n+1)})^2} \right\} \Big|_{s=0} \\
&= \frac{(1-q)q^n(nq^n(1-q)-q^{n+1})-q^n(1-q)(-1+(n+1)(1-q)q^n)}{(1-q)^2 q^{2n}} \\
&= \frac{n(1-q)q^n - q^{n+1} + 1 - (n+1)(1-q)q^n}{(1-q)^n q^n} \\
&= \frac{1-q^n}{q^n(1-q)}.
\end{aligned}
$$

Note that for $n = 1$, this equation reduces to $\mathbf{E}[T] = 1/q$, which is the mean of a geometrically-distributed random variable, as expected.

5. We calculate $f_{X|Y}(x|y)$ using the definition of a conditional density. To find the density of Y, recall that Y is normal, so the mean and variance completely specify $f_Y(y)$. $Y = X + N$, so $\mathbf{E}[Y] = \mathbf{E}[X] + \mathbf{E}[N] = 0 + 0 = 0$. Because X and N are independent, $\text{var}(Y) = \text{var}(X) + \text{var}(N) = \sigma_x^2 + \sigma_n^2$. So,

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{f_X(x) f_N(y-x)}{f_Y(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma_x^2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{x^2}{2\sigma_x^2} - \frac{(y-x)^2}{2\sigma_n^2}}}{\frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_n^2)}} e^{-\frac{y^2}{2(\sigma_x^2+\sigma_n^2)}}} \\
&= \frac{1}{\sqrt{2\pi \frac{\sigma_x^2 \sigma_n^2}{\sigma_x^2+\sigma_n^2}}} e^{\frac{y^2}{2(\sigma_x^2+\sigma_n^2)} - \frac{x^2}{2\sigma_x^2} - \frac{(y-x)^2}{2\sigma_n^2}}.
\end{aligned}
$$

We can simplify the exponent as follows.

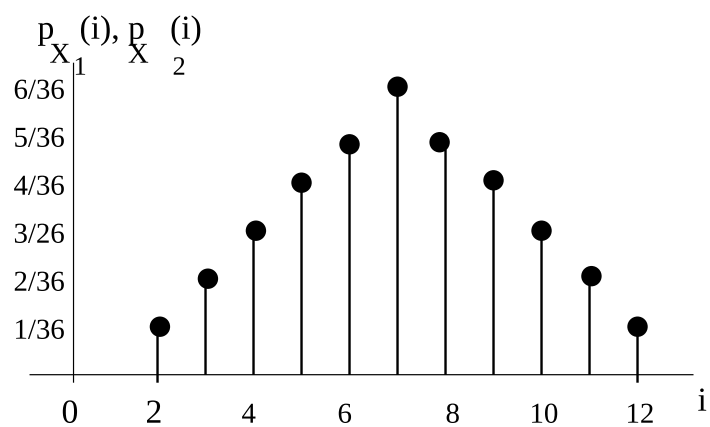
$$
\begin{aligned}
& \frac{y^2}{2(\sigma_x^2+\sigma_n^2)} - \frac{x^2}{2\sigma_x^2} - \frac{(y-x)^2}{2\sigma_n^2} \\
&= \frac{\sigma_x^2+\sigma_n^2}{2\sigma_x^2\sigma_n^2} \left(\frac{y^2\sigma_x^2\sigma_n^2}{(\sigma_x^2+\sigma_n^2)^2} - \frac{x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2} - \frac{(y-x)^2\sigma_x^2}{\sigma_x^2+\sigma_n^2} \right) \\
&= \frac{\sigma_x^2+\sigma_n^2}{2\sigma_x^2\sigma_n^2} \left(\frac{y^2\sigma_x^2\sigma_n^2 - x^2\sigma_n^2(\sigma_n^2+\sigma_x^2) - (y-x)^2\sigma_x^2(\sigma_x^2+\sigma_n^2)}{(\sigma_x^2+\sigma_n^2)^2} \right) \\
&= \frac{\sigma_x^2+\sigma_n^2}{2\sigma_x^2\sigma_n^2} \left(\frac{y^2\sigma_x^2\sigma_n^2 - x^2\sigma_n^4 - x^2\sigma_x^2\sigma_n^2 - y^2\sigma_x^4 - y^2\sigma_x^2\sigma_n^2 - x^2\sigma_x^4 - x^2\sigma_x^2\sigma_n^2 + 2xy\sigma_x^4 + 2xy\sigma_x^2\sigma_n^2}{(\sigma_x^2+\sigma_n^2)^2} \right) \\
&= \frac{\sigma_x^2+\sigma_n^2}{2\sigma_x^2\sigma_n^2} \left(\frac{-y^2\sigma_x^4 - x^2(\sigma_x^4 + 2\sigma_x^2\sigma_n^2 + \sigma_n^4) + 2xy(\sigma_x^4 + \sigma_x^2\sigma_n^2)}{(\sigma_x^2+\sigma_n^2)^2} \right) \\
&= -\frac{\sigma_x^2+\sigma_n^2}{2\sigma_x^2\sigma_n^2} \left(x - y\frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2} \right)^2.
\end{aligned}
$$

Thus, we obtain

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi \frac{\sigma_x^2 \sigma_n^2}{\sigma_x^2+\sigma_n^2}}} e^{-\frac{\left(x-y\frac{\sigma_x^2}{\sigma_x^2+\sigma_n^2}\right)^2}{\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}}}.$$

Looking at this formula, we see that the conditional density is normal with mean $\frac{\sigma_x^2 y}{\sigma_x^2+\sigma_n^2}$ and variance $\frac{\sigma_x^2\sigma_n^2}{\sigma_x^2+\sigma_n^2}$.

6. Let R_i be the number rolled on the i^{th} die. Since each number is equally likely to rolled, the PMF of each R_i is uniformly distributed from 1 to 6. The PMF of X_1 is obtained by convolving the PMFs of R_1 and R_2. Similarly, the PMF of X_2 is obtained by convolving the PMFs of R_3 and R_4. X_1 and X_2 take on values from 2 to 12 and are independent and identically distributed random variables. The PMF of either one is given by



Note that the sum $X_1 + X_2$ takes on values from 4 to 24. The discrete convolution formula tells us that for n from 4 to 24:

$$P(X_1 + X_2 = n) = \sum_{i=1}^{n} P(X_1 = i)P(X_2 = n - i)$$

so

$$P(X_1 + X_2 = 8) = \sum_{i=1}^{8} P(X_1 = i)P(X_2 = 8 - i)$$

and thus we find the desired probability is $\frac{35}{36^2} = .027$.

7. The PDF for X and Y are as follows,

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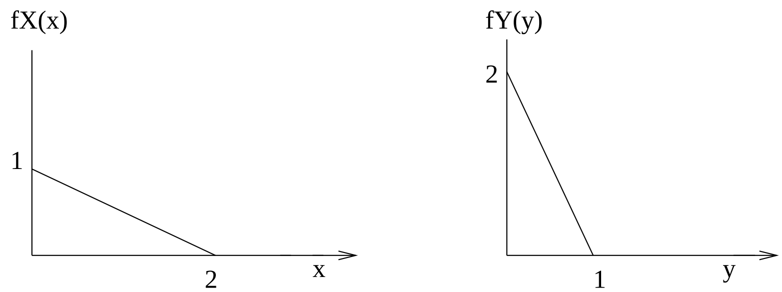
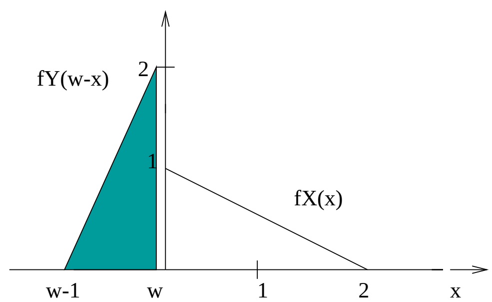


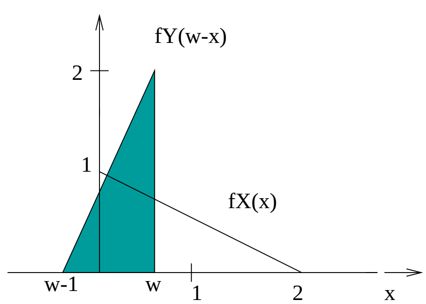
Because X and Y are independent and $W = X + Y$, the pdf of W, $f_W(w)$, can be written as the convolution of $f_X(x)$ and $f_Y(y)$:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) dx$$

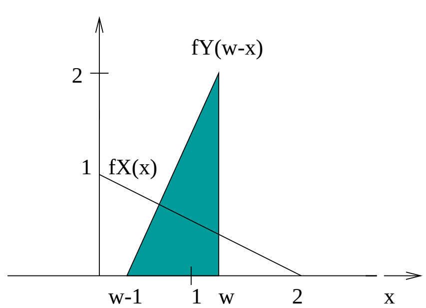
There are five ranges for w: 1. $w \leq 0$



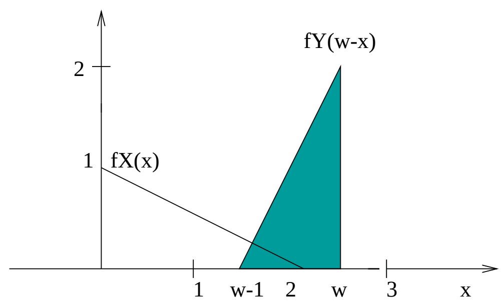
2. $0 \leq w \leq 1$



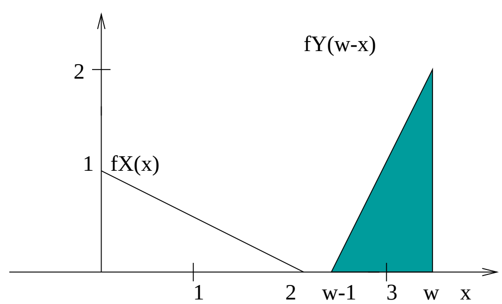
3. $1 \leq w \leq 2$



4. $2 \leq w \leq 3$



5. $3 \leq w$



$$f_W(w) = \begin{cases} \int_0^w f_X(x) f_Y(w-x)dx, & 0 \leq w \leq 1 \\ \int_{w-1}^w f_X(x) f_Y(w-x)dx, & 1 \leq w \leq 2 \\ \int_{w-1}^2 f_X(x) f_Y(w-x)dx, & 2 \leq w \leq 3 \\ 0, & otherwise \end{cases}$$

Therefore,

$$f_W(w) = \begin{cases} 2w - \frac{3}{2}w^2 + \frac{1}{6}w^3, & 0 \leq w \leq 1 \\ \frac{7}{6} - \frac{1}{2}w, & 1 \leq w \leq 2 \\ \frac{9}{2} - \frac{9}{2}w + \frac{3}{2}w^2 - \frac{1}{6}w^3, & 2 \leq w \leq 3 \\ 0, & otherwise \end{cases}$$

G1†. To compute $f_W(w)$, we will start by computing the joint PDF $f_{Y,Z}(y,z)$. Computing the joint density is quite simple. Define the joint CDF $F_{Y,Z}(y,z) = \mathbf{P}(Y \leq y, Z \leq z)$. Now, $F_Z(z) = \mathbf{P}(Z \leq z) = z^n$, because the maximum is less than z if and only if every one of the X_i is less than z, and all the X_i's are independent. We can also compute $\mathbf{P}(y \leq Y, Z \leq Z) = (z-y)^n$ because the minimum is greater than y and the maximum is less that z if and only if every X_i falls between y and z. Subtraction gives

$$F_{Y,Z}(y,z) = z^n - (z-y)^n.$$

Now, we find the joint PDF by differentiating, which gives $f_{Y,Z}(y,z) = n(n-1)(z-y)^{n-2}, 0 \leq y \leq z \leq 1$. Because Y and Z are not independent, convolving the individual densities for Y and Z will not give us the density for W. Instead, we must calculate the CDF $F_W(w)$ by integrating $P_{Y,Z}(y,z)$ over the appropriate region. We consider the cases $w \leq 1$ and $w > 1$ separately.

When $w \leq 1$, we need to compute

$$\int_0^{\frac{w}{2}} \int_y^{w-y} f_{Y,Z}(y,z)dzdy = \frac{w^n}{2}.$$

When $w > 1$, we can compute the CDF from

$$1 - \int_{\frac{w}{2}}^1 \int_{w-z}^{z} f_{Y,Z}(y,z)dydz = 1 - \frac{(2-w)^n}{2}.$$

Finally, we take the derivative to get

$$f_W(w) = \begin{cases} n\frac{w^{n-1}}{2} & ; \quad 0 \leq w \leq 1 \\ n\frac{(2-w)^{n-1}}{2} & ; \quad 1 \leq w \leq 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

To prove the concentration result, it is easier to look at $F_W(w)$. The CDF is exponential in n. Thus, $\mathbf{P}(W \leq 1-\epsilon) = \frac{(1-\epsilon)^n}{2}$ and $\mathbf{P}(W \geq 1+\epsilon) = 1 - (1 - \frac{(2-(1+\epsilon))^n}{2}) = \frac{(1-\epsilon)^n}{2}$. It is easily seen that both of these probabilities go to 0 as $n \to \infty$, which proves the desired concentration result.