15.084J Recitation Handout 10

Tenth Week in a Nutshell:

- Importance of duality
- Lagrangian dual approach
- Features of the Dual
- Column-geometry dual approach
- Weak Duality
- Strong Duality

Importance of Duality

- In many problems, dual variables have useful interpretations; current/voltage in electrical networks, consumption/prices in economic models, tensions/displacements in statics, etc
- The dual gives bounds on the achievable values of the primal – lets you know when your approximation is "good enough", lets you do branch and bound techniques, etc.
- Optimal dual solution is a certificate of optimality of primal.
- Often has nicer structure than the primal, so might be easier to work with.

Lagrangian Duality

- Begin with the primal: $\min f(x)$ st $g(x) \leq 0, x \in X$
- Take annoying constraints into objective with multipliers: $L(u) = f(x) + u^T g(x)$
- Now minimize L for the given u: $L^*(u) = min_{x \in X} f(x) + u^T g(x)$
- Assuming that was an easy minimization, we now maximize that over all positive u: $v^* = min_{u \geq 0} L^*(u)$

Features of the Dual

- Any feasible solution to the dual is a lower bound on the primal optimum.
- Many nice problems have nice duals: LP's stay LP's, QP's stay QP's, and log-barriers stay log-barriers.
- Many ugly problems have nice duals (usually with duality gaps).

Column Geometry Approach

- View the constraints as having a cost associated with them.
- Consider the set $I(s, z)$: there exists an X with objective better than z, and cost better than s.
- All solutions to the primal with $s \leq 0$ correspond to points on the z-axis, and indeed, the optimial is the lowest point on the z axis in I.
- The dual now is to find the supporting hyperplane with the highest z-intercept. This is obviously a lower bound on the optimal; if I is nice, it is a tight lower bound.
- If X and $g_i(x)$ are convex, I is convex (and thus nice).
- If you grind through the algebra, the support hyperplane problem is exactly the lagrangian dual.

Weak Duality

A dual solution is a lower bound on any primal solution. Dual optimum at negative infinity equates to no feasible dual solution. Dual optimum at positive infinity equates to no feasible primal solution.

Saddle Points

A point $\bar{x}, \bar{u}$ is a saddlepoint if for any x and any $u \geq 0$, $L(\bar{x}, u) \leq L(\bar{x}, \bar{u}) \leq L(x, \bar{u})$.

A point is a saddle point iff they are primal and dual optimal with no duality gap.

Convexity gives us KKT equivalent to saddle point; this means convex problems never have duality gaps!

Strong Duality

We want to find conditions under which there is no duality gap (also known as strong duality).

- Assume X open and convex, f convex, g convex

- Consider the perturbation function $z(y)$ which is the original problem with the RHS of the constraints replaced by y.
- This is continuous and convex where it exists
- If there is a subgradient at zero, then strong duality holds!