BEH. 410 . 03/17/2003
L# 12.

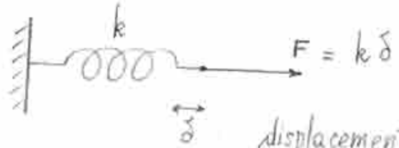
Last time : linear elastic tissue properties
composition & structure of ECM — collagen (tensile stiffness)
elastin (stretchy)
proteoglycans (compressive stiffness)

Today : equilibrium equations (handout)
strain energy density (and examples)
viscoelastic (time - dependent) behavior

- Elastic energy - the " strain energy density "

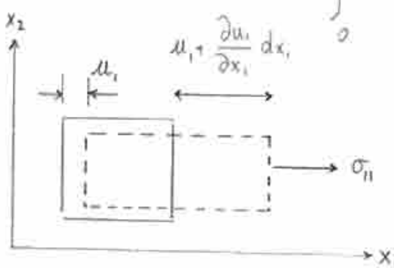
U = total strain energy

spring at equilibrium



Work done converted into stored energy : δ displacement of end

$$W = U = \int_0^{\delta} F\,dx = \int_0^{\delta} k\delta\, d\delta = \boxed{\frac{1}{2} k \delta^2 = U}$$



where $\frac{\partial u_1}{\partial x_1} dx_1$ is the relative displacement (equivalent to δ)

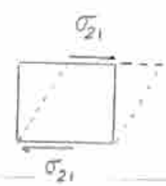
$$dW = dU = \int_0^{\Delta u_1} \sigma_{11}\, d\left(\frac{\partial u_1}{\partial x_1} dx_1\right) dx_2\, dx_3 = \int_0^{\Delta u_1} \sigma_{11}\, d\underbrace{\left(\frac{\partial u_1}{\partial x_1}\right)}_{\varepsilon_{11}} \underbrace{dx_1 dx_2 dx_3}_{d\forall \text{ volume of element}}$$

$$dU = \int_0^{\varepsilon_{11}} E\, \tilde{\varepsilon}_{11}\, d\tilde{\varepsilon}_{11}\, d\forall$$

$\hookrightarrow$ $\tilde{\varepsilon}_{11}$ dumb variable $\neq \varepsilon_{11}$

$$\frac{dU}{d\forall} = \boxed{\begin{aligned} &\frac{1}{2} E\, \varepsilon_{11}^2 = U_0 \\ =\ &\frac{1}{2}\sigma_{11}\varepsilon_{11} = \frac{1}{2}\frac{\sigma_{11}^2}{E}\end{aligned}}$$ strain energy density

- In a pure shear experiment $\boxed{U_0 = \sigma_{21}\,\varepsilon_{21} = \frac{\sigma_{21}^2}{2G} = 2G\,\varepsilon_{21}^2}$



- More generally in 3D

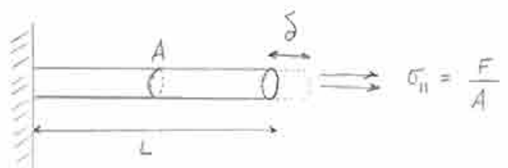
$$\boxed{\begin{aligned} U_0 &= \frac{1}{2}\left\{ \lambda (\varepsilon_{kk})^2 + 2G\,\varepsilon_{kk}^2 + 4G\left(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2\right)\right\} \\ &= \frac{1}{2E}\sigma_{kk}^2 - \frac{\nu}{E}\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}\right) + \frac{1}{2G}\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2\right)\end{aligned}}$$

- Scaling (more useful here) $\boxed{U_0 \sim G\,\varepsilon^2 \quad \text{or} \quad U_0 \sim \frac{\sigma^2}{G}}$

$\forall_{def}$: volume over which deformation occurs

Total strain energy $U = \int_{\forall} U_0\, d\forall \simeq U_0\, \forall_{def}$

▷ Example : uniform unidirectional extension



$\sigma_{22} = \sigma_{33} = 0$

no shear stresses

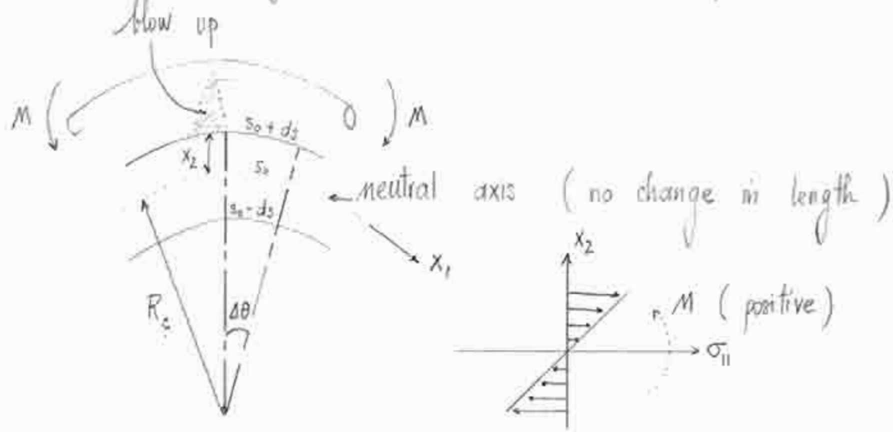
hence $U_0 = \frac{\sigma_{11}^2}{2E}$

$$U = \int_{\forall} U_0 \, d\forall = AL \frac{\sigma_{11}^2}{2E} = \frac{F^2 L}{2AE}$$

Work done $= \frac{1}{2} F \delta =$ stored elastic energy $= \frac{F^2 L}{2AE}$

$\frac{\delta}{L} = \varepsilon_{11} = \frac{F}{AE} = \frac{\sigma_{11}}{E}$ result from constitutive law

▷ Example 2: <u>bending stiffness</u> : uniform bending of a thin rod



$\varepsilon_{11} = \frac{ds}{s_0} = \frac{x_2}{R_c}$ from

$$\begin{cases} ds + s_0 = (R_c + x_2)\,\Delta\theta \\ s_0 = R_c \, d\theta \\ ds = x_2 \, d\theta \end{cases}$$

$-M = \int \sigma_{11} x_2 \, dA$

$\sigma_{11} = k x_2$ from drawing ; hence

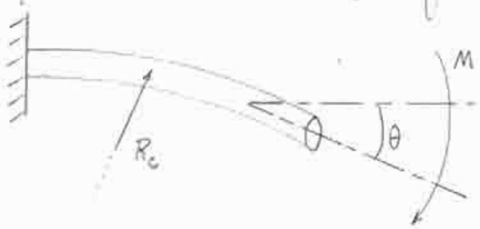
$-M = \int_A k x_2^2 \, dA = k I$

$k = -\frac{M}{I}$ ← moment of inertia

$\sigma_{11} = -\frac{M x_2}{I}$ and $U = \int_{\forall} \frac{\sigma_{11}^2}{2E} \, d\forall = \int_{\forall} \frac{M^2 x_2^2}{2E I^2} \, dx_1 dA$

$$\boxed{U = \int_0^L \frac{M^2}{2EI} \, dx_1 = \frac{M^2 L}{2EI}}$$

Bending of cantelevered rod (of length L)



$\frac{1}{2} M\theta = \frac{M^2 L}{2EI}$

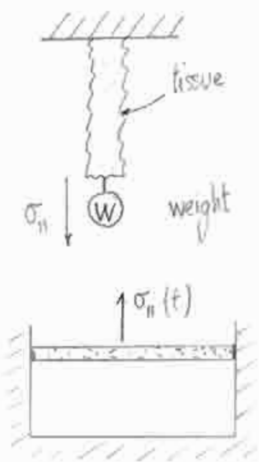
$\theta = \frac{L}{R_c} = \frac{ML}{EI}$

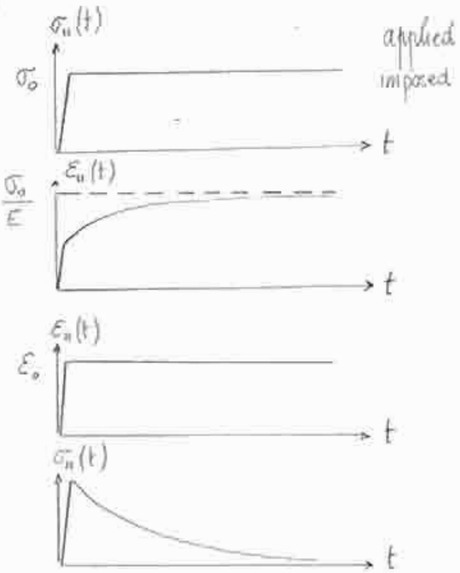
κ_f flexural rigidity
$\qquad\qquad$ ॥
K_B bending stiffness $= EI$

} hence $M = \frac{\kappa_f}{R_c}$

▫ <u>Linear viscoelastic (time - dependent) behavior</u>

▷ Example experiment



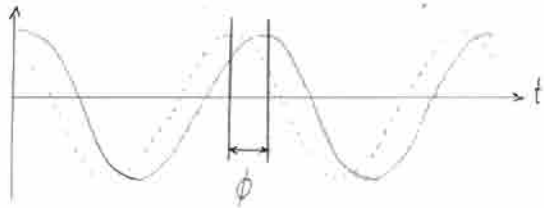


① <u>creep</u>

② <u>stress relaxation</u>

③ Dynamic, oscillatory testing

$\varepsilon_{11}(t)$ — $\sigma_{11}(t)$ ----



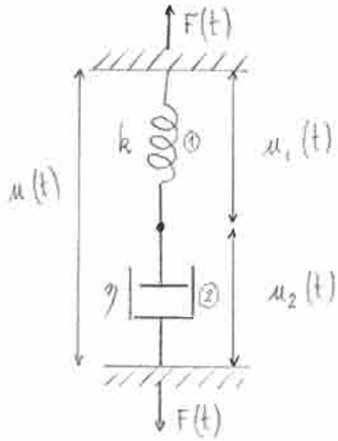
$\varepsilon_{11} = \varepsilon_0 \cos(\omega t)$ applied

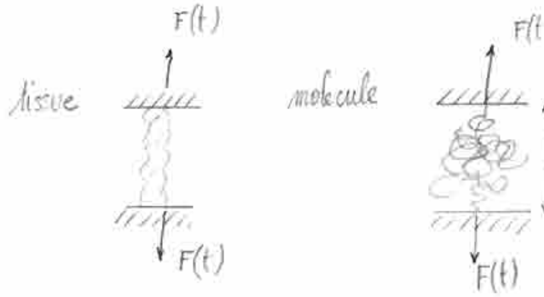
σ_{11} = result

stress leads strain by phase ϕ

- Lumped parameter model: (representation)

<u>Maxwell model</u>





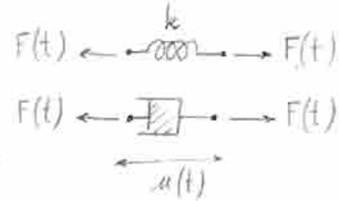
{ spring of constant k
{ dashpot of constant η

origin of viscoelastic behavior
- ability of microstructure to reorient or remodel (active process)
- expulsion of liquid from tissue
- internal viscous interactions

* analysis of Maxwell's model

spring element

dashpot element



elastic $F = k u$

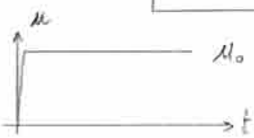
viscous $F = \eta \dfrac{du}{dt}$

$$\begin{cases} F_1 = k\, u_1 \\ F_2 = \eta \dfrac{du_2}{dt} \\ F = F_1 = F_2 \end{cases} \qquad \begin{cases} u = u_1 + u_2 \\ \dfrac{du}{dt} = \dfrac{1}{k}\dfrac{dF}{dt} + \dfrac{F}{\eta} \end{cases}$$

$$\boxed{k \frac{du}{dt} = \frac{dF}{dt} + F \frac{k}{\eta}}$$

Maxwell model

→ stress relaxation : find $F(t)$ given



$$F(t) = A \exp\left(-\frac{t}{\tau_R}\right) + B$$

$\hookrightarrow = 0$ because $F \xrightarrow[t \to +\infty]{} 0$

at $t = 0^+$ $F = F_0 = k\, u_0$

$\tau_R = \eta / k$

$$\boxed{F(t) = k\, u_0 \exp\left(-\frac{t}{\tau_R}\right)}$$