# Notes for Recitation 16

**Problem 1.** Find closed-form generating functions for the following sequences. Do not concern yourself with issues of convergence.

**(a)** $\langle 2, 3, 5, 0, 0, 0, 0, \ldots \rangle$

**Solution.**

$$2 + 3x + 5x^2$$

**(b)** $\langle 1, 1, 1, 1, 1, 1, 1, \ldots \rangle$

**Solution.**

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

**(c)** $\langle 1, 2, 4, 8, 16, 32, 64, \ldots \rangle$

**Solution.**

$$\begin{aligned} 1 + 2x + 4x^2 + 8x^3 + \ldots &= (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \ldots \\ &= \frac{1}{1-2x} \end{aligned}$$

**(d)** $\langle 1, 0, 1, 0, 1, 0, 1, 0, \ldots \rangle$

**Solution.**

$$1 + x^2 + x^4 + x^6 + \ldots = \frac{1}{1-x^2}$$

**(e)** $\langle 0, 0, 0, 1, 1, 1, 1, 1, \ldots \rangle$

**Solution.**

$$x^3 + x^4 + x^5 + x^6 + \ldots = x^3(1 + x + x^2 + x^3 + \ldots) \qquad = \frac{x^3}{1-x}$$

**(f)** $\langle 1, 3, 5, 7, 9, 11, \ldots \rangle$

**Solution.**

$$
\begin{aligned}
1 + x + x^2 + x^3 + \ldots &= \frac{1}{1-x} \\
\frac{d}{dx}\, 1 + x + x^2 + x^3 + \ldots &= \frac{d}{dx}\, \frac{1}{1-x} \\
1 + 2x + 3x^2 + 4x^2 + \ldots &= \frac{1}{(1-x)^2} \\
2 + 4x + 6x^2 + 8x^2 + \ldots &= \frac{2}{(1-x)^2} \\
1 + 3x + 5x^2 + 7x^3 + \ldots &= \frac{2}{(1-x)^2} - \frac{1}{1-x} \\
&= \frac{1+x}{(1-x)^2}
\end{aligned}
$$

**Problem 2.** Find a closed-form generating function for the sequence

$$(t_0, t_1, t_2, \ldots)$$

where $t_n$ is the number of different ways to compose a bag of $n$ donuts subject to the following restrictions.

**(a)** All the donuts are chocolate and there are at least 3.

**Solution.**

$$\frac{x^3}{1-x}$$

**(b)** All the donuts are glazed and there are at most 4.

**Solution.**

$$\frac{1-x^5}{1-x}$$

**(c)** All the donuts are coconut and there are exactly 2.

**Solution.**

$$x^2$$

**(d)** All the donuts are plain and the number is a multiple of 4.

**Solution.**

$$\frac{1}{1-x^4}$$

**(e)** The donuts must be chocolate, glazed, coconut, or plan and:

- There must be at least 3 are chocolate donuts.
- There must be at most 4 glazed.
- There must be exactly 2 coconut.
- There must be a multiple of 4 plain.

**Solution.**

$$\frac{x^3}{1-x}\frac{1-x^5}{1-x}x^2\frac{1}{1-x^4} = \frac{x^5(1+x^2+x^3+x^4)}{(1-x)(1-x^4)}$$

**Problem 3.** [20 points] A previous problem set introduced us to the Catalan numbers: $C_0, C_1, C_2, \ldots$, where the $n$-th of them equals the number of balanced strings that can be built with $2n$ paretheses. Here is a list of the first several of them:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_n$ | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 | 4862 | 16796 | 58786 | 208012 |

Then, in lecture we were all amazed to see that the decimal expansion of the irrational number $500000 - 1000\sqrt{249999}$

$$1.0000010000020000050000140000420001320004290014300048620167960587862080 12\ldots$$

"encodes" these numbers! Now, there are many reasons why one may want to turn to religion, but this revelation is probably not a good one. Let's explain why.

**(a)** Let $p_n$ be the number of balanced strings containing $n$ ('s. Explain why the following recurrence holds:

$$p_0 = 1, \qquad \text{(the empty string)}$$

$$p_n = \sum_{k=1}^{n} p_{k-1} \cdot p_{n-k}, \qquad \text{for } n \geq 1.$$

**Solution.** Note that every nonempty balanced string consists of a sequence of one or more balanced strings. The first balanced string in the sequence must begin with a ( and end with a "matching" ). That is, any balanced string, $r_n$, with $n \geq 1$ ('s consists of a balanced string, $s_{k-1}$, enclosed in brackets and containing $k-1$ ('s, followed by a balanced string, $t_{n-k}$, with $n-k$ ('s:

$$r_n = (s_{k-1}) \text{ followed by } t_{n-k}$$

where $1 \leq k \leq n$. This observation leads directly to the recurrence.

**(b)** Now consider the generating function for the number of balanced strings:

$$P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots.$$

Prove that

$$P(x) = xP(x)^2 + 1.$$

**Solution.** We can verify this equation using the recurrence relation.

$$\begin{aligned}
xP(x)^2 + 1 &= x(p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots)^2 + 1 \\
&= x(p_0^2 + (p_0 p_1 + p_1 p_0)x + (p_0 p_2 + p_1 p_1 + p_2 p_0)x^2 + \cdots) + 1 \\
&= 1 + p_0^2 x + (p_0 p_1 + p_1 p_0)x^2 + (p_0 p_2 + p_1 p_1 + p_2 p_0)x^3 + \cdots \\
&= 1 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \\
&= P(x)
\end{aligned}$$

**(c)** Find a closed-form expression for the generating function $P(x)$.

**Solution.** Given that $P(x) = xP(x)^2 + 1$, the quadratic formula implies that

$$P(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

If $x$ is small, then $P(x)$ should be about $p_0 = 1$. Therefore, the correct choice of sign is

$$P(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

**(d)** Show that $P(1/1000000) = 500000 - 1000\sqrt{249999}$.

**Solution.**

$$\begin{aligned} P(1/1000000) &= \frac{1 - \sqrt{1 - 4/1000000}}{2/1000000} \\ &= 500000 - 500000\sqrt{\frac{249999}{250000}} \\ &= 500000 - 1000\sqrt{249999} \end{aligned}$$

**(e)** Explain why the digits of this irrational number encode these successive numbers of balanced strings.

**Solution.** Suppose that we symbolically carry out the substitution done in the preceding problem part.

$$\begin{aligned} P(x) &= p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots \\ P(10^{-6}) &= p_0 + p_1 10^{-6} + p_2 10^{-12} + p_3 10^{-18} + \cdots \end{aligned}$$

Thus, $p_0$ appears in the units position, $p_1$ appears in the millionths position, $p_2$ appears in the trillionths position, and so forth.

**Problem 4.** Consider the following recurrence equation:

$$T_n = \begin{cases} 1 & n = 0 \\ 2 & n = 1 \\ 2T_{n-1} + 3T_{n-2} & (n \geq 2) \end{cases}$$

Let $f(x)$ be a generating function for the sequence $\langle T_0, T_1, T_2, T_3, \ldots \rangle$.

**(a)** Give a generating function in terms of $f(x)$ for the sequence:

$$\langle 1, \quad 2, \quad 2T_1 + 3T_0, \quad 2T_2 + 3T_1, \quad 2T_3 + 3T_2, \ldots \rangle$$

**Solution.** We can break this down into a linear combination of three sequences:

$$\begin{array}{llllllll} \langle & 1, & 2, & 0, & 0, & 0, & \ldots & \rangle & = 1 + 2x \\ \langle & 0, & T_0, & T_1, & T_2, & T_3, & \ldots & \rangle & = xf(x) \\ \langle & 0, & 0, & T_0, & T_1, & T_2, & \ldots & \rangle & = x^2 f(x) \end{array}$$

In particular, the sequence we want is very nearly generated by $1 + 2x + 2xf(x) + 3x^2 f(x)$. However, the second term is not quite correct; we're generating $2 + 2T_0 = 4$ instead of the correct value, which is $2$. We correct this by subtracting $2x$ from the generating function, which leaves:

$$1 + 2xf(x) + 3x^2 f(x)$$

**(b)** Form an equation in $f(x)$ and solve to obtain a closed-form generating function for $f(x)$.

**Solution.** The equation

$$f(x) = 1 + 2xf(x) + 3x^2 f(x)$$

equates the left sides of all the equations defining the sequence $T_0, T_1, T_2, \ldots$ with all the right sides. Solving for $f(x)$ gives the closed-form generating function:

$$f(x) = \frac{1}{1 - 2x - 3x^2}$$

**(c)** Expand the closed form for $f(x)$ using partial fractions.

**Solution.** We can write:

$$1 - 2x - 3x^2 = (1 + x)(1 - 3x)$$

Thus, there exist constants $A$ and $B$ such that:

$$f(x) = \frac{1}{1 - 2x - 3x^2} = \frac{A}{1 + x} + \frac{B}{1 - 3x}$$

Now substituting $x = 0$ and $x = 1$ gives the system of equations:

$$1 = A + B$$
$$-\frac{1}{4} = \frac{A}{2} - \frac{B}{2}$$

Solving the system, we find that $A = 1/4$ and $B = 3/4$. Therefore, we have:

$$f(x) = \frac{1/4}{1+x} + \frac{3/4}{1-3x}$$

**(d)** Find a closed-form expression for $T_n$ from the partial fractions expansion.

**Solution.** Using the formula for the sum of an infinite geometric series gives:

$$f(x) = \frac{1}{4}\left(1 - x + x^2 - x^3 + x^4 - \ldots\right) + \frac{3}{4}\left(1 + 3x + 3^2x^2 + 3^3x^3 + 3^4x^4 + \ldots\right)$$

Thus, the coefficient of $x^n$ is:

$$T_n = \frac{1}{4} \cdot (-1)^n + \frac{3}{4} \cdot 3^n$$