Mathematics for Computer Science

MIT 6.042J/18.062J

State Machines II: Derived Variables, Stable Marriage

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Derived Variables

A *derived variable, v,* is a function giving a "value" to each state:

$v: Q \to$ Values.

If Values = $\mathbb{N}$, we'd say v was

"natural-number-valued," or "$\mathbb{N}$-valued."

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Derived Variables

Robot on the grid example:

States $Q = \mathbb{N}^2$.

Define the sum-value, σ, of a state:

$$\sigma(\langle x,y \rangle) ::= x+y$$

An $\mathbb{N}$-valued derived variable.

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Derived Variables

Another derived variable:

$$\pi ::= \sigma \pmod 2.$$

π is $\{0,1\}$-valued.

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Derived Variables

Called "derived" to distinguish from *actual* variables that appear in a program.

For robot Actual: x, y

Derived: σ, π

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Derived Variables

For GCD, have (actual) variables x, y.

Proof of GCD termination:

y is strictly decreasing and natural number-valued.

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Derived Variables

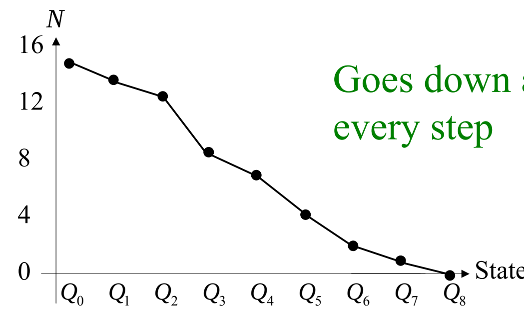
Termination followed by

Well Ordering Principle:

y must take a least value –

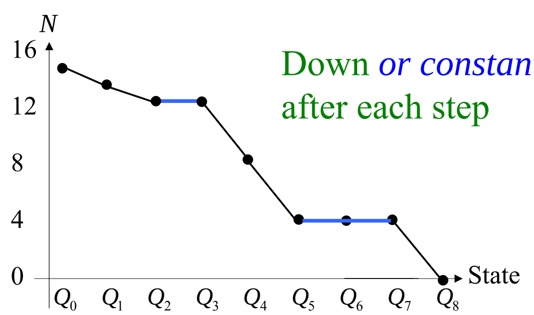
and then the algorithm is stuck.

Strictly Decreasing Variable



Goes down at every step

Weakly Decreasing Variable



Down *or constant* after each step

σ, π for the *Diagonal Robot*

σ: up & down all over the place –

neither increasing nor decreasing.

π: is constant –

both increasing & decreasing

(weakly)

Weakly Decreasing Variable

(We used to call weakly decreasing variables "nonincreasing" variables.)

Team Problem

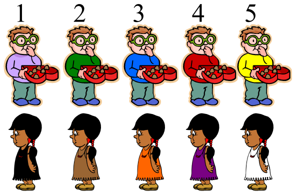
Problem 1

Stable Marriage

A Marriage Problem

1 2 3 4 5

Boys



Girls

A B C D E

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Stable Marriage

Preferences:

Boys	Girls
1: CBEAD	A : 35214
2 : ABECD	B : 52143
3 : DCBAE	C : 43512
4 : ACDBE	D : 12345
5 : ABDEC	E : 23415

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Stable Marriage

Preferences

1: CBEAD
2 : ABECD
3 : DCBAE
4 : ACDBE
5 : ABDEC

Try “greedy” strategy for boys

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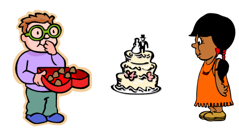
Stable Marriage

Preferences

~~1: CBEAD~~
2 : ABE~~C~~D
3 : D~~C~~BAE
4 : A~~C~~DBE
5 : ABDE~~C~~

Marry Boy 1 with Girl C (his 1st choice)



1 C

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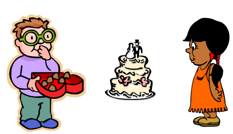
Stable Marriage

Preferences

2 : ABE D
3 : D BAE
4 : A DBE
5 : ABDE

Marry Boy 1 with Girl C (his 1st choice)



1 C

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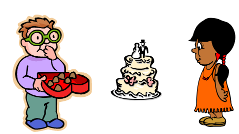
Stable Marriage

Preferences

2 : ABED
3 : DBAE
4 : ADBE
5 : ABDE

Marry Boy 1 with Girl C (his 1st choice)



1 C

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Stable Marriage

Preferences

1 : C

~~2 : ABED~~

3 : DB~~A~~E

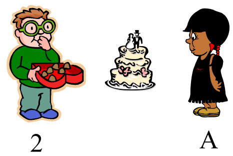
4 : ~~A~~DBE

5 : ~~A~~BDE

Next:

Marry Boy 2 with Girl A:

(best remaining choice)



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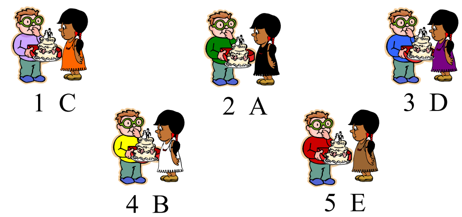
Stable Marriage

Final "boy greedy" marriages



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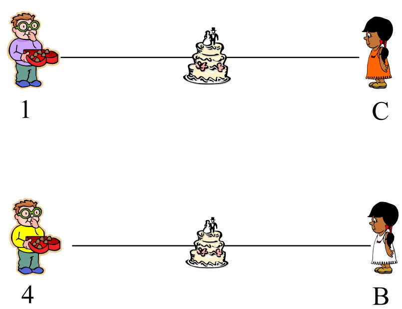
Stable Marriage

Trouble!



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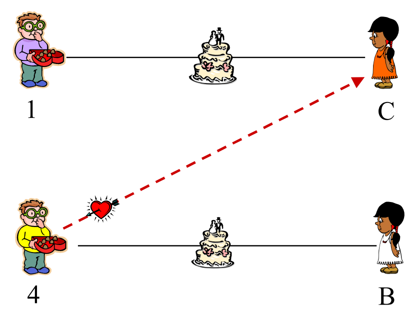
Stable Marriage

Boy 4 likes Girl C better than his wife.



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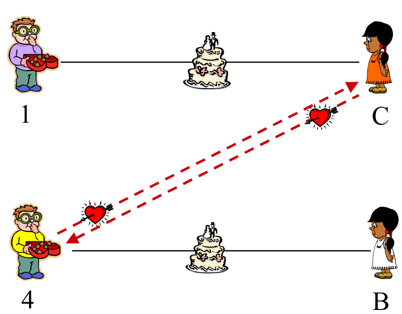
Stable Marriage

and vice-versa



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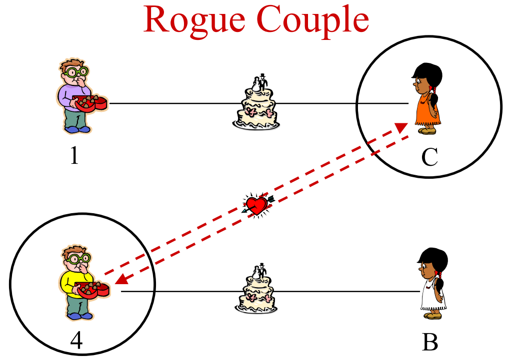
Stable Marriage

Rogue Couple



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Stable Marriage

Stable Marriage Problem:

Marry everyone without any rogue couples!

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Stable Marriage

Let’s Try it!

?Volunteers:

5 Boys & 5 Girls

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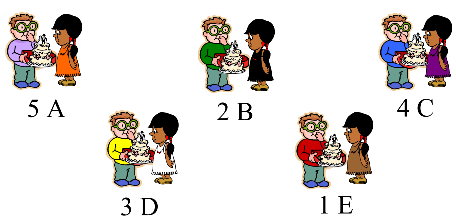
Stable Marriage I.



Boy Optimal

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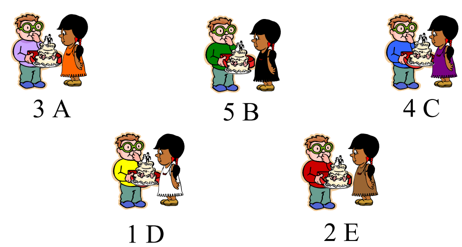
Stable Marriage II.



All Girls get 1st Choice

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Stable Marriage

More than a puzzle:

- College Admissions

  (original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.
- Matching Dance Partners.

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Stable Marriage

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Stable Marriage

The Mating Algorithm:

day by day

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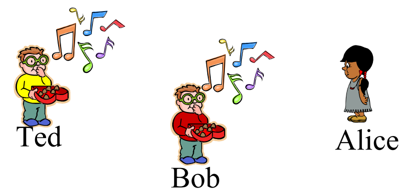
Mating Algorithm

Morning: boy serenades favorite girl



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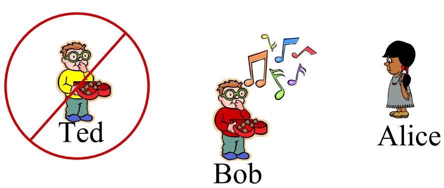
Mating Algorithm

Morning: boy serenades favorite girl

Afternoon: girl rejects all but favorite



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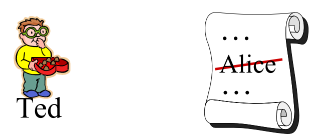
Mating Algorithm

Morning: boy serenades favorite girl

Afternoon: girl rejects all but favorite

Evening: rejected boy writes off girl



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Mating Algorithm

Stop when no girl rejects.

Girl marries her favorite suitor.

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Team Problem

Problem 2

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Mating Algorithm

Partial Correctness:

- Everyone is married.
- Marriages are stable.

Termination:

there exists a Wedding Day.

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Mating Algorithm

Model as State Machine

State q:

Each boy's set of $\underbrace{\text{"eligible"}}_{\text{not crossed off}}$ girls

$q(\text{Bob}) = \{\text{Carole, Alice, } \ldots\}$

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Mating Algorithm: variables

Derived Variable

serenading(Bob):

Bob's favorite eligible girl.

("Top" on Bob's list.)

$::= \max\{q(\text{Bob})\}$

using Bobs' preference order.

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Mating Algorithm: variables

Derived Variable

suitors(Alice):

all boys serenading Alice.

$::= \textit{serenading}^{-1}(\text{Alice})$

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Stable Marriage: Termination

Derived Variable

boy's-list-length:

total number of names not crossed off boy's lists

$::= \Sigma_{b \,\in\, \text{boys}} |q(b)|$

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Stable Marriage: Termination

boy's-list-length:

strictly decreasing & $\mathbb{N}$-valued.

So $\exists$ Wedding Day.

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Mating Algorithm: variables

Derived Variable

favorite(Carole):
Carole's preferred suitor.
::= max{*suitors*(Carole)}
using Carole's preference order.

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Mating Algorithm

Different girls have different favorites,
because boys serenade
one girl at a time.
(*favorite*: Girls→Boys
is an injection)

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Mating Algorithm: Girls improve

Lemma: A girl's favorite tomorrow will be at least as desirable as today's.

That is, *favorite*(*G*) is weakly increasing for each *G*.

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Mating Algorithm: Girls improve

Lemma: A girl's favorite tomorrow will be at least as desirable as today's.

…because today's favorite will stay until she rejects him for someone better.

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Mating Algorithm: Boys Get Worse

Lemma: A boy's 1st love tomorrow will be no more desirable than today's.

That is, *serenading*(*B*) is weakly decreasing for each *B*.

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Mating Algorithm: Boys Get Worse

Lemma: A boy's 1st love tomorrow will be no more desirable than today's.

…because boys work straight down their lists.

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Mating Algorithm: Invariant

If G has rejected B, then she has a better current favorite.

Proof:

favorite(G) is weakly increasing.

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Stable Marriage: Termination

On Wedding Day:

- Each girl has $\leq$ 1 suitors
- Each boy is married, or has no girls on his list

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Mating Algorithm: Everyone Marries

Everyone is Married by Wedding Day

Proof: by contradiction.

If B is not married, his list is empty.

By Invariant, all girls have favorites *better* than B -- so they *do* have a favorite.

That is, all girls are married.

So all boys are married.

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Team Problem

Problem 3

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Mating Algorithm

Who does better, boys or girls?

Girls' suitors get better, and boy's sweethearts get worse, so girls do better? No!

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Mating Algorithm

Mating Algorithm is *Optimal* for all Boys at once. Pessimal for all Girls.

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Stable Marriage

More questions, rich theory:

Other stable marriages possible?
- Can be many.

Can a boy do better by lying? – No!
Can a girl do better by lying? – Yes!

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