

Solutions to In-Class Problems Week 5, Mon.

Problem 1. There is a party. Some people shake hands an even number of times and some shake an odd number of times. Show that an even number of people shake hands an odd number of times.

Solution. We can represent the party by a graph. (Yeah, right.) Each person is represented by a vertex. If two people shake hands, then there is an edge between the corresponding vertices. This reduces the problem to the following theorem:

Theorem 1.1. *In every graph, there are an even number of vertices of odd degree.*

Proof. Partitioning the vertices into those of even degree and those of odd degree, we know

$$\sum_{v \in V} d(v) = \sum_{d(v) \text{ is odd}} d(v) + \sum_{d(v) \text{ is even}} d(v)$$

The value of the lefthand side of this equation is even, and the second summand on the righthand side is even since it is entirely a sum of even values. So the first summand on the righthand side must also be even. But since it is entirely a sum of odd values, it must contain an even number of terms. That is, there must be an even number of vertices with odd degree. \square

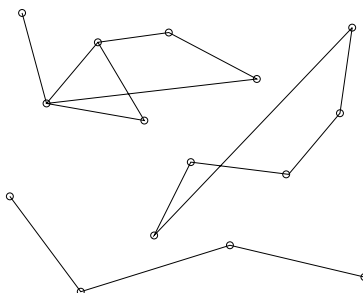
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Problem 2. Define a *halfgraph* to be a graph with $2n$ vertices, all of degree exactly n , for some $n \geq 1$. Can a halfgraph have more than one connected component? If so, draw an example. If not, explain why.

Solution. No. If the graph has two or more connected components, then some connected component must have n or fewer vertices. (Otherwise, there would be more than $2n$ vertices in all.) Each vertex in this component can have degree at most $n - 1$. \blacksquare

Problem 3. A set, M , of vertices of a graph is a *maximal connected set* if every pair of vertices in the set are connected, and any set of vertices properly containing M will contain two vertices that are not connected.

(a) What are the maximal connected subsets of the following (unconnected) graph?



Solution. There are three maximal subsets, each one equal to the vertices of one of the connected components of the graph. ■

(b) Explain the connection between maximal connected sets and connected components.

Solution. They are the same. ■

(c) Prove it.

Solution. We first show that a connected component is a maximal set. A connected component consists of *all* the vertices connected to some vertex v . A larger set of vertices would, by definition, contain both v and a vertex, w , that is not in the connected component. This means that w is not connected to v , and therefore the larger set is not connected. So a connected component is maximal.

Conversely, suppose we have a maximal connected set, M . Since M is connected, any vertex, v , in M is connected to all the other vertices in M . If there was any vertex, w , connected to v , that was not in M , then $M \cup \{w\}$ would be connected and properly contain M , contradicting the maximality of M . So M consists of exactly the vertices connected to v , proving that it is a connected component. ■

Problem 4. (a) Describe a connected graph such that every vertex is on a simple cycle, but the graph is not 2-connected.

Solution. Two cycles connected by an edge. ■

