Solutions to In-Class Problems Week 14, Mon.

Problem 1. Here are seven propositions:

$$\begin{array}{rcrcr} x_1 & \vee & x_3 & \vee & \neg x_7 \\ \neg x_5 & \vee & x_6 & \vee & x_7 \\ x_2 & \vee & \neg x_4 & \vee & x_6 \\ \neg x_4 & \vee & x_5 & \vee & \neg x_7 \\ x_3 & \vee & \neg x_5 & \vee & \neg x_8 \\ x_9 & \vee & \neg x_8 & \vee & x_2 \\ \neg x_3 & \vee & x_9 & \vee & x_4 \end{array}$$

Note that:

1. Each proposition is the OR of three terms of the form x_i or the form $\neg x_i$.
2. The variables in the three terms in each proposition are all different.

Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability.

(a) What is the probability that a single proposition is true?

Solution. Each proposition is true unless all three of its terms are false. Thus, each proposition is true with probability:

$$1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

■

(b) What is the expected number of true propositions?

Solution. Let T_i be an indicator for the event that the i-th proposition is true. Then the number of true propositions is $T_1 + \ldots + T_7$ and the expected number is:

$$\begin{aligned} \mathrm{E}[T_1 + \ldots + T_7] &= \mathrm{E}[T_1] + \ldots + \mathrm{E}[T_7] \\ &= 7/8 + \ldots + 7/8 \\ &= 49/8 = 6\frac{1}{8} \end{aligned}$$

■

(c) Use your answer to prove that there exists an assignment to the variables that makes *all* of the propositions true.

Solution. A random variable can not always be less than its expectation, so there must be some assignment such that:

$$T_1 + \ldots T_7 \geq 6\frac{1}{8}$$

This implies that $T_1 + \ldots + T_7 = 7$ for at least one outcome. This outcome is an assignment to the variables such that all of the propositions are true. ■

Problem 2. Final exams in 6.042 are graded according to a rigorous procedure:

- With probability $4/7$ the exam is graded by a *recitation instructor*, with probability $2/7$ it is graded by a *lecturer*, and with probability $1/7$, it is accidentally dropped behind the radiator and arbitrarily given a score of 84.
- *Recitation instructors* score an exam by scoring each problem individually and then taking the sum.
  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability $3/4$, and no credit is given with probability $1/4$.
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.
- *Lecturers* score an exam by rolling a fair die twice, multiplying the results, and then adding a "general impression" score.

- With probability $4/10$, the general impression score is 40.
- With probability $3/10$, the general impression score is 50.
- With probability $3/10$, the general impression score is 60.

Assume all random choices during the grading process are mutually independent.

(a) What is the expected score on an exam graded by a recitation instructor?

Solution. Let X equal the exam score and C be the event that the exam is graded by a recitation instructor. We want to calculate $\mathrm{E}[X \mid C]$. By linearity of (conditional) expectation, the expected sum of the problem scores is the sum of the expected problem scores. Therefore, we have:

$$\begin{aligned}
\mathrm{E}[X \mid C] &= 10 \cdot \mathrm{E}[\text{T/F score} \mid C] + 4 \cdot \mathrm{E}[\text{15pt prob score} \mid C] + \mathrm{E}[\text{20pt prob score} \mid C] \\
&= 10 \cdot \left(\frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0\right) + 4 \cdot \left(2 \cdot \frac{7}{2} + 3\right) + \left(\frac{1}{2} \cdot 12 + \frac{1}{2} \cdot 18\right) \\
&= 10 \cdot \frac{3}{2} + 4 \cdot 10 + 15 = 70
\end{aligned}$$

■

(b) What is the expected score on an exam graded by a lecturer?

Solution. Now we want $\mathrm{E}\left[X \mid \overline{C}\right]$, the expected score a lecturer would give. Employing linearity again, we have:

$$\begin{aligned}
\mathrm{E}\left[X \mid \overline{C}\right] &= \mathrm{E}\left[\text{product of dice} \mid \overline{C}\right] \\
&\quad + \mathrm{E}\left[\text{general impression} \mid \overline{C}\right] \\
&= \left(\frac{7}{2}\right)^2 && \text{(because the dice are independent)} \\
&\quad + \left(\frac{4}{10} \cdot 40 + \frac{3}{10} \cdot 50 + \frac{3}{10} \cdot 60\right) \\
&= \frac{49}{4} + 49 = 61\frac{1}{4}
\end{aligned}$$

■

(c) What is the expected score on a 6.042 exam?

Solution. Let X equal the true exam score. The Total Expectation Theorem implies:

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}[X \mid C]\Pr\{C\} + \mathrm{E}\left[X \mid \overline{C}\right]\Pr\left\{\overline{C}\right\} \\
&= 70 \cdot \frac{4}{7} + \left(\frac{49}{4} + 49\right) \cdot \frac{2}{7} + 84 \cdot \frac{1}{7} \\
&= 40 + \left(\frac{7}{2} + 14\right) + 12 = 69\frac{1}{2}
\end{aligned}$$

■

Problem 3. The number of squares that a piece advances in one turn of the game Monopoly is determined as follows:

- Roll two dice, take the sum of the numbers that come up, and advance that number of squares.
- If you roll *doubles* (that is, the same number comes up on both dice), then you roll a second time, take the sum, and advance that number of additional squares.
- If you roll doubles a second time, then you roll a third time, take the sum, and advance that number of additional squares.
- However, as a special case, if you roll doubles a third time, then you go to jail. Regard this as advancing zero squares overall for the turn.

(a) What is the expected sum of two dice, given that the same number comes up on both?

Solution. There are six equally-probable sums: 2, 4, 6, 8, 10, and 12. Therefore, the expected sum is:

$$\frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 4 + \ldots + \frac{1}{6} \cdot 12 = 7$$

■

(b) What is the expected sum of two dice, given that different numbers come up? (Use your previous answer and the Total Expectation Theorem.)

Solution. Let the random variables D_1 and D_2 be the numbers that come up on the two dice. Let E be the event that they are equal. The Total Expectation Theorem says:

$$\mathrm{E}[D_1 + D_2] = \mathrm{E}[D_1 + D_2 \mid E] \cdot \Pr\{E\} + \mathrm{E}\left[D_2 + D_2 \mid \overline{E}\right] \cdot \Pr\left\{\overline{E}\right\}$$

Two dice are equal with probability $\Pr\{E\} = 1/6$, the expected sum of two independent dice is 7, and we just showed that $\mathrm{E}[D_1 + D_2 \mid E] = 7$. Substituting in these quantities and solving the equation, we find:

$$\begin{aligned} 7 &= 7 \cdot \frac{1}{6} + \mathrm{E}\left[D_2 + D_2 \mid \overline{E}\right] \cdot \frac{5}{6} \\ \mathrm{E}\left[D_2 + D_2 \mid \overline{E}\right] &= 7 \end{aligned}$$

■

(c) To simplify the analysis, suppose that we always roll the dice three times, but may ignore the second or third rolls if we didn't previously get doubles. Let the random variable X_i be the sum of the dice on the i-th roll, and let E_i be the event that the i-th roll is doubles. Write the expected number of squares a piece advances in these terms.

Solution. From the total expectation formula, we get:

$$\begin{aligned} \mathrm{E}[\text{advance}] = {} & \mathrm{E}\left[X_1 \mid \overline{E_1}\right] \cdot \Pr\left\{\overline{E_1}\right\} \\ & + \mathrm{E}\left[X_1 + X_2 \mid E_1 \cap \overline{E_2}\right] \cdot \Pr\left\{E_1 \cap \overline{E_2}\right\} \\ & + \mathrm{E}\left[X_1 + X_2 + X_3 \mid E_1 \cap E_2 \cap \overline{E_3}\right] \cdot \Pr\left\{E_1 \cap E_2 \cap \overline{E_3}\right\} \\ & + \mathrm{E}[0 \mid E_1 \cap E_2 \cap E_3] \cdot \Pr\{E_1 \cap E_2 \cap E_3\} \end{aligned}$$

Then using linearity of (conditional) expectation, we refine this to

$$\begin{aligned} & \mathrm{E}[\text{advance}] \\ & = \mathrm{E}\left[X_1 \mid \overline{E_1}\right] \cdot \Pr\left\{\overline{E_1}\right\} \\ & \quad + \left(\mathrm{E}\left[X_1 \mid E_1 \cap \overline{E_2}\right] + \mathrm{E}\left[X_2 \mid E_1 \cap \overline{E_2}\right]\right) \cdot \Pr\left\{E_1 \cap \overline{E_2}\right\} \\ & \quad + \left(\mathrm{E}\left[X_1 \mid E_1 \cap E_2 \cap \overline{E_3}\right] + \mathrm{E}\left[X_2 \mid E_1 \cap E_2 \cap \overline{E_3}\right] + \mathrm{E}\left[X_3 \mid E_1 \cap E_2 \cap \overline{E_3}\right]\right) \\ & \quad \quad \cdot \Pr\left\{E_1 \cap E_2 \cap \overline{E_3}\right\} \\ & \quad + 0. \end{aligned}$$

Using mutual independence of the rolls, we simplify this to

$$\begin{aligned} & \mathrm{E}[\text{advance}] \\ & = \mathrm{E}\left[X_1 \mid \overline{E_1}\right] \cdot \Pr\left\{\overline{E_1}\right\} \qquad (1) \\ & \quad + \left(\mathrm{E}[X_1 \mid E_1] + \mathrm{E}\left[X_2 \mid \overline{E_2}\right]\right) \cdot \Pr\{E_1\} \cdot \Pr\left\{\overline{E_2}\right\} \\ & \quad + \left(\mathrm{E}[X_1 \mid E_1] + \mathrm{E}[X_2 \mid E_2] + \mathrm{E}\left[X_3 \mid \overline{E_3}\right]\right) \cdot \Pr\{E_1\} \cdot \Pr\{E_2\} \cdot \Pr\left\{\overline{E_3}\right\} \end{aligned}$$

■

(d) What is the expected number of squares that a piece advances in Monopoly?

Solution. We plug the values from parts (a) and (b) into equation (**??**):

$$\begin{aligned}
\mathrm{E}\left[\text{advance}\right] &= 7 \cdot \frac{5}{6} + (7+7) \cdot \frac{1}{6} \cdot \frac{5}{6} + (7+7+7) \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \\
&= 8\frac{19}{72}
\end{aligned}$$

■