PS#1 22.611J Answer Key

1) a) show

$$n = \int_{-\infty}^{\infty}\int\int n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m(v_x^2+v_y^2+v_z^2)}{2T}\right) d^3v$$

→ so we've (since $\exp(-a-b) = \exp^{-a}\exp^{-b}$)

$$n = n\left(\frac{m}{2\pi T}\right)^{3/2} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \exp\left(-\frac{m(v_x^2+v_y^2)}{2T}\right) \int_{-\infty}^{\infty} \exp\left(\frac{-mv_z^2}{2T}\right) dv_z$$

using $\frac{1}{2}\int_{-\infty}^{\infty} e^{-(ax^2)} dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^{\frac{1}{2}}$

and $a = \frac{m}{2T}$ w/ $x^2 = v_z^2$,

→ we've

$$\int_{-\infty}^{\infty} \exp\left(\frac{-mv_z^2}{2T}\right) dv_z = \left(\frac{2\pi T}{m}\right)^{\frac{1}{2}}$$

so now,

$$n = n\left(\frac{m}{2\pi T}\right) \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \exp\left(\frac{-m(v_x^2+v_y^2)}{2T}\right)$$

→ repeating the previous steps w/ v_x & v_y gives

$$n = n\left(\frac{m}{2\pi T}\right)\left(\frac{2\pi T}{m}\right)^{1/2}\left(\frac{2\pi T}{m}\right)^{1/2}$$

OR

→ $\boxed{n = n}$ where $n = n(x, y, z)$

1b) Now let's evaluate $\langle v_x \rangle$

$$\boxed{\langle v_x \rangle = \frac{\int d^3v\, v_x f(\vec{v})}{\int d^3v f(\vec{v})} = \frac{\int d^3v\, v_x f(\vec{v})}{n} = 0}$$

but since v_x is odd, & $f(\vec{v}) = f(v^2)$ is even, integrating from $-\infty$ to ∞ gives 0!

[($v_x f(v^2)$) is an odd function]!

→ makes sense, since $f(\vec{v})$ is a maxwellian centered at $v=0$

1c) find $\langle v^2 \rangle$

so we got

$$\langle v^2 \rangle = \frac{\int d^3v \; v^2 f(\vec{v})}{\left[\int d^3 v f(\vec{v}) = n \right]} = \left(\frac{m}{2\pi T}\right)^{3/2} \iiint_{d^3v} v^2 \exp\left(-\frac{mv^2}{2T}\right)$$

Let's switch over to spherical coordinates:

$$d^3v = v_r^2 \sin v_\theta \, dv_r \, d\theta \, d\phi$$

$$\langle v^2 \rangle = \left(\frac{m}{2\pi T}\right)^{3/2} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v_r^2 \exp\left(-\frac{m v_r^2}{2T}\right) v_r^2 \sin(v_\theta) \, dv_r \, d\theta \, d\phi$$

$$= \left(\frac{m}{2\pi T}\right)^{3/2} \int_0^{2\pi}\int_0^{\pi} dv_\theta \, dv_\phi \sin(v_\theta) \int_0^{\infty} v_r^4 \exp\left(-\frac{m v_r^2}{2T}\right) dv_r$$

$$= \left(\frac{m}{2\pi T}\right)^{3/2} \int_0^{2\pi}\int_0^{\pi} dv_\theta \, dv_\phi \sin(v_\theta) \left[\frac{\Gamma(5/2)}{2\left(\frac{m}{2T}\right)^{(5/2)}}\right]$$

since $\int_0^{\infty} x^m e^{-ax^2} dx = \dfrac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

and $\Gamma(5/2) = 1.5 \dfrac{\sqrt{\pi}}{2} = \dfrac{3}{4}\sqrt{\pi}$

$$\langle v^2 \rangle = \left(\frac{m}{2\pi T}\right)^{3/2} \int_0^{2\pi}\int_0^{\pi} dv_\theta \, dv_\phi \sin(v_\theta) \frac{3}{8}\left(\frac{2T}{m}\right)^{5/2} \sqrt{\pi}$$

$$= \underbrace{\frac{3}{8}\frac{2^{5/2}}{2^{3/2}}\left(\frac{\sqrt{\pi}}{\pi^{3/2}}\right)\left(\frac{T}{m}\right)}_{\frac{3}{4\pi}\frac{T}{m}} \int_0^{2\pi}\int_0^{\pi} dv_\theta \, dv_\phi \sin(v_\theta)$$

$$\langle v^2 \rangle = \frac{3}{4\pi}\frac{T}{m} \int_0^{2\pi} \Big[-\cos v_\theta\Big]_0^{\pi} dv_\phi$$

$$= \frac{3}{4\pi}\frac{T}{m} \int_0^{2\pi} 2 \, dv_\phi = \frac{3}{2\cancel{\pi}}\frac{T}{m}\cancel{2\pi}$$

$$= 3\frac{T}{m}$$

$$\boxed{\text{Hence, } \langle \tfrac{1}{2} m v^2 \rangle = \tfrac{3}{2} T}$$

1d) The average speed $\langle|v|\rangle$

Again, use spherical coordinates

$$\langle|v|\rangle = \frac{\int f(v)|v|\,dv}{n} = \iiint v_r \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m v_r^2}{2T}\right) dv_r\, d\theta\, d\phi\, v_r^2 \sin\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v_r^3 \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m v_r^2}{2T}\right) \sin\theta\, dv_r\, d\theta\, d\phi$$

using $\int_0^{\infty} x^m e^{-ax^2} dx = \frac{\Gamma\{(m+1)/2\}}{2a^{(m+1)/2}}$, and $\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = 4\pi$,

$$\langle|v|\rangle = 4\pi \left(\frac{m}{2\pi T}\right)^{3/2} \left[\frac{\Gamma\{(2)\}}{2\left(\frac{m}{2T}\right)^2}\right]$$

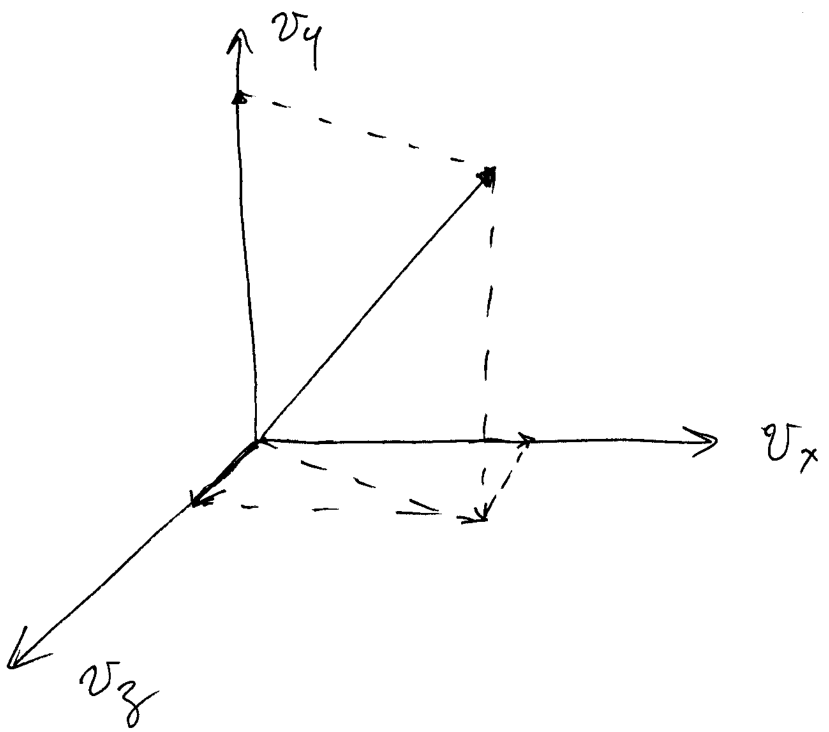
$$\langle|v|\rangle = \frac{\cancel{4}^{\,2}\pi}{\cancel{2}} \left(\frac{m}{2\pi T}\right)^{3/2} \left(\frac{2T}{m}\right)^2 \left[\Gamma\{(2\}\right], \quad \Gamma(2) = 1$$

$$= 2\frac{T^{1/2}}{m^{1/2}} \frac{\sqrt{2}}{\pi^{1/2}}$$

$$\boxed{\langle|v|\rangle = 2\sqrt{\frac{2T}{m\pi}} = \frac{2\sqrt{2}}{\sqrt{\pi}}\left(\frac{T}{m}\right) = \frac{2\sqrt{2}}{\sqrt{\pi}} v_{th}}$$

1 → Note on Velocity Space integrals (4)

- same as x, y, z, except replace w/ v_x, v_y, v_z



hence, in spherical coordinates:

$$v_r = (v_x^2 + v_y^2 + v_z^2)^{1/2} = \text{the speed!}$$

and since in space $dx\,dy\,dz$ in spherical coordinates is

$$dx\,dy\,dz = r^2 \sin\theta \, d\theta \, d\phi \, dr,$$

$$d^3v = v_r^2 \sin v_\theta \, dv_\theta \, dv_\phi \, dv_r$$

2) SI | CGS (5)

SI	CGS
a) $c = 3\times10^{8}\,\frac{m}{s}$	$c = 3\times10^{10}\,\frac{cm}{s}$
b) $e = \lvert q \rvert = 1.6\times10^{-19}\,C$	$e = 4.8\times10^{-10}\,esu$

c) $m_e c^2 = .511\ MeV$

d) $m_p c^2 = 940\ MeV$

e) $1eV \longrightarrow 11{,}600K$

f) $P = nkT$

$P = 1.01\times10^{5}\,Pa = n \cdot 1.38\times10^{-23}\,\frac{J}{K^{\circ}}\ 293.15K$

$\boxed{n = 2.5\times10^{19}\,\frac{\text{air molecules}}{cm^3}}$

e) $\rho_{H_2O} \sim 1000\,\frac{kg}{m^3}$ $\quad \frac{1000kg\ mole}{18\,gm} \rightarrow 56{,}000$ moles

$\rightarrow \frac{56{,}000\,moles}{m^3} \cdot 6.02\times10^{23}\,\frac{\#}{mole} = 3.35\times10^{28}\,\frac{H_2O}{m^3}$

$\rightarrow \boxed{3.35\times10^{22}\,\frac{H_2O}{cm^3} = \rho_{H_2O}}$

(h) $\boxed{E_I = \frac{1}{2}\,\frac{m_e c^2 e^4}{\hbar^2 c^2} = \frac{1}{2}\,.511\,MeV\left(\frac{1}{137}\right)^2 = 13.6\,eV}$

i) 10^4 gauss $= 1$ Tesla

j) $10^6\,cm^3 = 1\,m^3$

2) → CGS & SI System:

Maxwell's equations:

SI:

$$\vec{\nabla}\cdot\vec{D} = e(n_i - n_e)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{j} + \frac{\partial\vec{D}}{\partial t}$$

$$\vec{D} = \varepsilon_0\vec{E}, \quad \vec{B} = \mu_0\vec{H}$$

CGS

$$\vec{\nabla}\cdot\vec{E} = 4\pi e(n_i - n_e)$$

$$c\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$c\vec{\nabla}\times\vec{B} = 4\pi\vec{j} + \frac{\partial\vec{E}}{\partial t}$$

$$\epsilon = \mu = 1$$

Basic Quantities:

	mks(SI)	CGS
e	C (Coulomb)	esu (electrostatic unit)
$\vec{B}$	tesla	Gauss
$\vec{E}$	V/m	esu/cm
E/B	m/s	1
eV	1.6e-19 J	1.6e-12 erg

To go from cgs → mks (for most formulas)

- replace B/c by B
- 4π by ε_0^{-1}

(where $1/4\pi\varepsilon_0 = 9\times10^9$)

→ for more info check out NRL Formulary or Chen Appendix A

3) at 50% ionization, $n_e \sim n_i$ (quasi-neutrality)

$$P = \Sigma n k T = n_e k T + n_i k T + n_0 k T = 1.01 \times 10^5 Pa$$

where n_0 is the neutral density

so $n_e = n_i = n_0$, since 50% ionized

then,

$$n_0 = \frac{4}{(4\pi)^{5/2}} \left(\frac{m_e c^2}{\hbar c} \frac{e^2}{\hbar c} \right)^3 \left(\frac{T}{E_I} \right)^{3/2} \exp\left(-\frac{E_I}{T} \right)$$

$$= \frac{4.85 \times 10^{22}}{cm^3} \left(\frac{T}{13.6} \right)^{3/2} \exp\left(-\frac{13.6}{T} \right)$$

using $\frac{1.01 \times 10^5 Pa}{3} = n_0 T$ (from $P = \Sigma n k T$)

OR $2.1 \times 10^{17} \frac{eV}{cm^3} = n_0 T$

$$\frac{2.1 \times 10^{17} eV/cm^3}{T} = \frac{4.85 \times 10^{22}}{cm^3} \left(\frac{T}{13.6} \right)^{3/2} \exp\left(\frac{-13.6}{T} \right)$$

4.33×10^{-6}

$$\boxed{T = 1.45 eV}$$

4) In general,

$$f_{max}(\vec{x},\vec{v},t) = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\left(\frac{\text{Particle energy}}{\text{temp}}\right)\right)$$

so if $\phi(r) \neq 0$,

$$f(\vec{x},\vec{v},t) = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{\left(\frac{mv^2}{2} + q\phi\right)}{T}\right)$$

then,

$$n = \iiint f\, d^3v = n_0 \exp\left(\frac{-q\phi}{T}\right)$$

where $n_0 = n_{i\infty} = n_{e\infty}$ (far away from potential)

Using Maxwell's equations:

$$\varepsilon_0 \vec{\nabla}^2 \phi = \rho \qquad \text{since} \quad \vec{E} = \vec{\nabla}\phi$$
$$= e(n_i - n_e)$$

also, since $T \gg q\phi$,

$$n \simeq n_0\left(1 - \frac{q\phi}{T}\right)$$

then,

$$\vec{\nabla}^2\phi = +\frac{e}{\varepsilon_0}\left(-n_0\left(1 - \frac{e\phi}{T_i}\right) + n_0\left(1 + \frac{e\phi}{T_e}\right)\right)$$

$$= \frac{e^2 n_0}{\varepsilon_0}\left(\frac{\phi}{T_e} + \frac{\phi}{T_i}\right) = \frac{e^2 n_0 \phi}{\varepsilon_0}\left(\frac{1}{T_e} + \frac{1}{T_i}\right)$$

call $D = \left(\frac{1}{T_e} + \frac{1}{T_i}\right)$

then,

$$\vec{\nabla}^2\phi = \frac{e^2 n_0 \phi}{\varepsilon_0} D$$

4) Con't

but $\frac{e^2 n_0}{\varepsilon_0} D \ [=] \ \frac{1}{\text{length}^{1/2}}$

call this $\frac{1}{\lambda_{ie}^2} \Rightarrow \lambda_{ie}^2 = \frac{\varepsilon_0}{e^2 n_0 D}$

then,

$$\vec{\nabla}^2 \phi = \frac{\phi}{\lambda_{ie}^2}$$

In spherical coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$$

substitute $u = \phi \cdot r$ into $\vec{\nabla}^2 \phi = \frac{\phi}{\lambda_{ie}^2}$

and we get $\frac{\partial^2 u}{\partial r^2} = \frac{u}{\lambda_{ie}^2}$

$$\frac{\partial^2 u}{\partial r^2} - \frac{u}{\lambda_{ie}^2} = 0$$

~~general~~ try soln: $u = A e^{-\alpha r}$

$$\Rightarrow (-\alpha)^2 \cancel{e^{-\alpha r}} - \frac{\cancel{e^{-\alpha r}}}{\lambda_{ie}^2} = 0$$

$$\alpha = \left(\frac{1}{\lambda_{ie}^2}\right)^{\frac{1}{2}} = \pm \frac{1}{\lambda_{ie}}$$

so,

general soln: $u = A e^{\frac{-r}{\lambda_{ie}}} + B e^{\frac{+r}{\lambda_{ie}}}$

hence, $\phi = \frac{A e^{-r/\lambda_{ie}}}{r} + \frac{B e^{r/\lambda_{ie}}}{r}$

$\rightarrow$ B.C as $r \to \infty$, $\phi \to 0$,

so $B = 0$

as $r \to 0$, $\phi = \frac{q}{4\pi\varepsilon_0 r}$

so $\phi = \frac{q}{4\pi\varepsilon_0 \cancel{r}} = \frac{A}{\cancel{r}}$ at $r \to 0$

$A = \frac{q}{4\pi\varepsilon_0}$

4) con't

$$\phi = \frac{q}{4\pi\varepsilon_0 r} e^{-r/\lambda_{ie}} \quad \text{where} \quad \lambda_{ie} = \left(\frac{\varepsilon_0}{e^2 n_0 \left(\frac{1}{T_e} + \frac{1}{T_i} \right)} \right)^{\frac{1}{2}}$$

(in SI units)

$$\phi = \frac{q}{r} e^{-r/\lambda_{ie}} \quad w/ \quad \lambda_{ie} = \left(\frac{1}{4\pi n e^2 \left(\frac{1}{T_e} + \frac{1}{T_i} \right)} \right)^{\frac{1}{2}}$$

(in cgs)

→ letting $T_i \to 0$ and $T_e \gg T_i$, we get

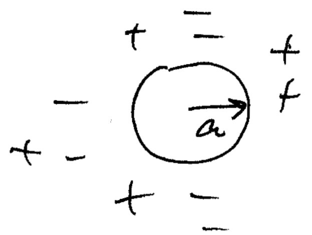
$$\phi \simeq 0 \qquad \text{since } \lambda_{ie} \to 0$$

→ so we don't get the 'immobile' ion approximation by letting $T_i \to 0$!!!

→ why? at $T_i = 0$, it means the ions 've no thermal velocity / energy to damp out responses to the field → not that they're immobile!

→ the ions 'll move to shield out the charge entirely!

5)



$n_e = n_0 \exp(e\phi/T_e)$

$n_i = n_0$

(n_0 is value far away from potential)

a) Lets use gauss law again

$$\nabla^2\phi = \frac{\rho}{\varepsilon_0} = -\frac{n_0}{\varepsilon_0}\left(1 - \left(1 + \frac{e\phi}{T_e}\right)\right)e$$

since $n_e \sim n_0\left(1 + \frac{e\phi}{T_e}\right)$

$$\nabla^2\phi + \frac{n_0}{\varepsilon_0}\left(-\frac{e^2\phi}{T_e}\right) = 0$$

$$\nabla^2\phi - \frac{\phi}{\lambda^2} = 0$$

In spherical coordinates,

$$\nabla^2 \rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)$$

as in problem 4), this gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) - \frac{\phi}{\lambda^2} = 0$$

using $U = \phi \cdot r$ substitution, we get

$$\phi = \frac{A}{r}e^{-r/\lambda_D} + \frac{B}{r}e^{r/\lambda_D}$$

using our B.C.s

$\therefore$ $r \rightarrow \infty$, $\phi \rightarrow 0$, here $B = 0$

$r \rightarrow a$, $\phi \rightarrow \phi_s$

OR

$$\phi_s = \frac{A}{a}e^{-\frac{a}{\lambda_D}} \qquad A = \frac{a\phi_s}{e^{-a/\lambda_D}} = a\phi_s e^{a/\lambda_D}$$

5a) Con't

so,

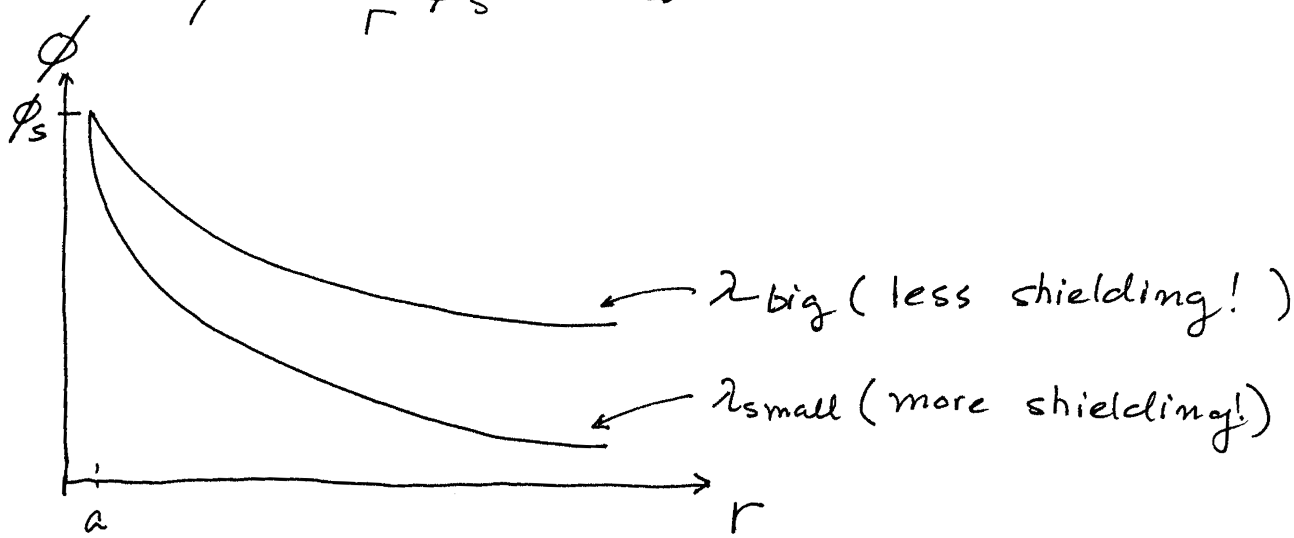
$$\boxed{\phi = \frac{a \phi_s e^{a/\lambda_D} e^{\frac{-r}{\lambda_D}}}{r}}$$

5b) w/ $\lambda_D \gg a$, we've

$$\phi = \frac{a}{r}\phi_s e^{\left(\frac{a}{\lambda_D} - \frac{r}{\lambda_D}\right)} \approx \frac{a}{r}\phi_s e^{\left(\frac{-r}{\lambda_D}\right)}$$

w/ $\lambda_D \ll a$,

$$\phi = \frac{a}{r}\phi_s e^{\left(\frac{a-r}{\lambda_D}\right)}$$



(i.e. if the debye length is long, the more thermal energy the electrons have. Hence they stick around less to shield the central charge).

5c) We can use the following to calculate the total charge:

$$\oint \vec{E} \cdot d\vec{A} = \frac{1}{\varepsilon_0} \int_{v} \rho_f \, dv$$

and $\bar{E} = -\bar{\nabla}\phi$

$$\bar{E} = -\frac{\partial \phi}{\partial r} = -\left(\frac{-a}{r^2} e^{a/\lambda_D} \phi_s e^{-r/\lambda_D} + \frac{a}{r}\phi_s e^{a/\lambda_D}\left(-\frac{1}{\lambda_D}\right) e^{-r/\lambda_D}\right)$$

$$= \frac{a}{r^2} e^{a/\lambda_D} \phi_s \left(e^{-r/\lambda_D} + \frac{r}{\lambda_D} e^{-r/\lambda_D} \right)$$

$$= \frac{a}{r^2} e^{a/\lambda_D} \phi_s e^{-r/\lambda_D} \left(1 + \frac{r}{\lambda_D} \right)$$

hence

$$4\pi a^2 E = \frac{Q_{sphere}}{\varepsilon_0}$$

$$Q_s = \varepsilon_0 4\pi a^2 \left(\frac{\cancel{a}}{\cancel{a^2}} \cancel{e^{a/\lambda_D}} \phi_s \cancel{e^{-a/\lambda_D}} \right)\left(1 + \frac{a}{\lambda_D} \right)$$

$$Q_s = \varepsilon_0 4\pi a \phi_s \left(1 + \frac{a}{\lambda_D} \right)$$

defining $C = \dfrac{Q}{V} = \varepsilon_0 \dfrac{4\pi a \cancel{\phi_s} \left(1 + \frac{a}{\lambda_D}\right)}{\cancel{\phi_s} - \underbrace{\phi_\infty}_{=0}}$

$$\boxed{\begin{array}{ll} C = 4\pi a \left(1 + \frac{a}{\lambda_D}\right)\varepsilon_0 & (\text{in SI}) \quad [=] \text{ Faraday} \\ C = a\left(1 + \frac{a}{\lambda_D}\right) & (\text{in cgs} \quad [=] \text{ cm}) \end{array}}$$

5d) so if $\lambda_D \gg a$,

$$C = 4\pi\varepsilon_0 a (1)$$

if $\lambda_D \ll a$

$$C = 4\pi\varepsilon_0 \frac{a^2}{\lambda_D}$$

i) $n_0 = 10^{14}/cm^3$, $T = 1\,keV$, $a = 10\,cm$

$\lambda_D = 2.35 \times 10^{-3}\,cm$

$C = 4.7 \times 10^{-8}\,F$ (in SI)

$C = 42{,}621\,cm$ (in cgs)

ii) $n_0 = 10^6/cm^3 \Rightarrow \lambda = 23.4\,cm$

$C = 1.6 \times 10^{-11}\,F$ (in SI)

$C = 14.3\,cm$ (in cgs)

→ the vacuum capacitance is simply

$$C = \frac{Q}{V}$$

$$\phi_s = \frac{Q}{4\pi\varepsilon_0 a} \Rightarrow$$

$$C = \frac{\cancel{\phi_s} 4\pi\varepsilon_0 a}{\cancel{\phi_s}}$$

$= 4\pi\varepsilon_0 a = 1.11 \times 10^{-11}\,F$ (in SI)

$= 10\,cm = a$ (in cgs)

so, the effect of short λ_D/ high e^- density is to act as a dielectric! (since

$\boxed{C_i \gg C_{ii} \sim C_{vacuum}}$