Massachusetts Institute of Technology

Subject 6.651J/8.613J/22.611J R. Parker
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Problem Set 3 (continued)

Problem 1 (iii). Hint: You may find integral **3.131-**8 in “Table of Integrals, Series and Products” by Gradshteyn and Ryzhik useful (several copies are on reserve in the PSFC library).

Problem 2.

The problem here is to calculate how a high energy beam of ions slows down by colliding with the electrons and ions in a plasma. Consider a mono-energetic beam of ions with mass m_b , charge $Z_b e$, and velocity $\vec{v}_b = \hat{x} v_b$. The beam slows down by means of Coulomb collisions with the plasma in accordance with the relation

$$m_b \frac{dv_b}{dt} = -m_b (\nu_{be} + \nu_{bi}) v_b$$

Here, ν_{be} and ν_{bi} are the $90\,°$ beam-electron and beam-ion momentum loss collision frequencies respectively. When the beam has lost all of its directed velocity, all of its initial kinetic energy has been transferred to the plasma.

The value of ν_{bi}, (with all the correct coefficients) follows from our analysis in class and is given by

$$\nu_{bi}(v_b) = \frac{1}{4\pi} \frac{Z_b^2 e^4 n_i}{\varepsilon_0^2 m_r m_b v^3} \ln\Lambda \approx \frac{1}{4\pi} \frac{Z_b^2 e^4 n_i}{\varepsilon_0^2 m_r m_b v_b^3} \ln\Lambda = 0.94 \frac{n_{20}}{(m_b v_b^2/2)^{3/2}} \ \text{sec}^{-1}$$

Here we have assumed that the thermal velocity of the plasma ions is much less than the beam velocity. Thus, the relative velocity between beam ions and plasma ions is accurately approximated by $v \approx v_b$. The numerical value corresponds to 3.5 MeV alpha particles slowing down on 15 keV ions in a D-T fusion reactor. Also n_{20} is the density measured in units of 10^{20} particles/m^3 and $m_b v_b^2/2$ is measured in MeV. The quantity $\ln\Lambda = 20$. .

A similar relationship exists for ν_{be} except that in this case, because of the small electron mass, the electron thermal velocity is much greater than the beam velocity. Therefore, the average relative velocity between electrons and beam ions is approximately equal to the electron thermal velocity: $v \approx v_{Te}$. A careful calculation that gets all the numerical coefficients correct is given by

$$\nu_{be}(v_b) = \frac{1}{3(2\pi)^{3/2}} \frac{Z_b^2 e^4 m_e^{1/2} n_e}{\varepsilon_0^2 m_b T_e^{3/2}} \ln\Lambda = 100 \frac{n_{20}}{T_e^{3/2}} \ \text{sec}^{-1}$$

where T_e is the plasma temperature measured in keV.

a.) Observe that for large values of v_b the electron collision frequency is larger than the ion collision frequency. For small v_b the ions dominate. The critical transition velocity v_{crit} occurs when $\nu_{be} = \nu_{bi}$. Derive an analytic formula for v_{crit} and numerically evaluate the ratio $m_b v_{crit}^2 / 2T_e$ assuming that $m_b = 4m_{proton}$, corresponding to alpha particles, and $m_i = 2.5 m_{proton}$, corresponding to a 50-50 fuel mix. If the alphas start with an energy of 3.5 MeV and the electrons are also at $T_e = 15$ keV, at what beam energy does the transition occur?

b. Solve the differential equation to determine $v_b(t)$ assuming that $v_b(0)$ corresponds to 3.5 MeV. Calculate the time t_{crit} for the beam ions to slow down to the transition value. During this time most of the energy has been transferred to electrons. Calculate how long it takes for the remainder of the beam energy to transfer mainly to the ions: $\Delta t = t_{final} - t_{crit}$ with t_{final} the time where the beam has slowed down to the ion thermal velocity $m_b v_b^2 / 2 \approx T$. At this time the beam has essentially lost all of its energy. Assume $n_{20} = 2$.

c. The last part of the problem involves calculating what fraction of the total beam energy is transferred to electrons and what fraction to ions. Define the alpha energy as $U_b = m_b v_b^2 / 2$. Conservation of energy then implies that

$$\frac{dU_e}{dt} = 2\nu_{be} U_b$$
$$\frac{dU_i}{dt} = 2\nu_{bi} U_b$$

Evaluate U_e and U_i by integrating these equations using the solution for $U_b(t)$ obtained in part (b). Integrate from $t = 0$ until $t = t_{final}$ and evaluate the energy fractions

$$f_e = \frac{U_e}{U_e + U_i}$$
$$f_i = \frac{U_i}{U_e + U_i}$$

at $t = t_{final}$. At $T = 15$ keV, what fraction of alpha energy is transferred to electrons and what fraction to ions? The integrals can be evaluated numerically, but with a little thought several accurate approximations can be made that allow a fully analytic solution.