Lecture 17
Other Eigenvalue Algorithms

MIT 18.335J / 6.337J

Introduction to Numerical Methods

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November 7, 2006

The Jacobi Algorithm

- Diagonalize 2×2 real symmetric matrix by a *Jacobi rotation*:

$$J^T \begin{bmatrix} a & d \\ d & b \end{bmatrix} J = \begin{bmatrix} \neq 0 & 0 \\ 0 & \neq 0 \end{bmatrix}$$

where

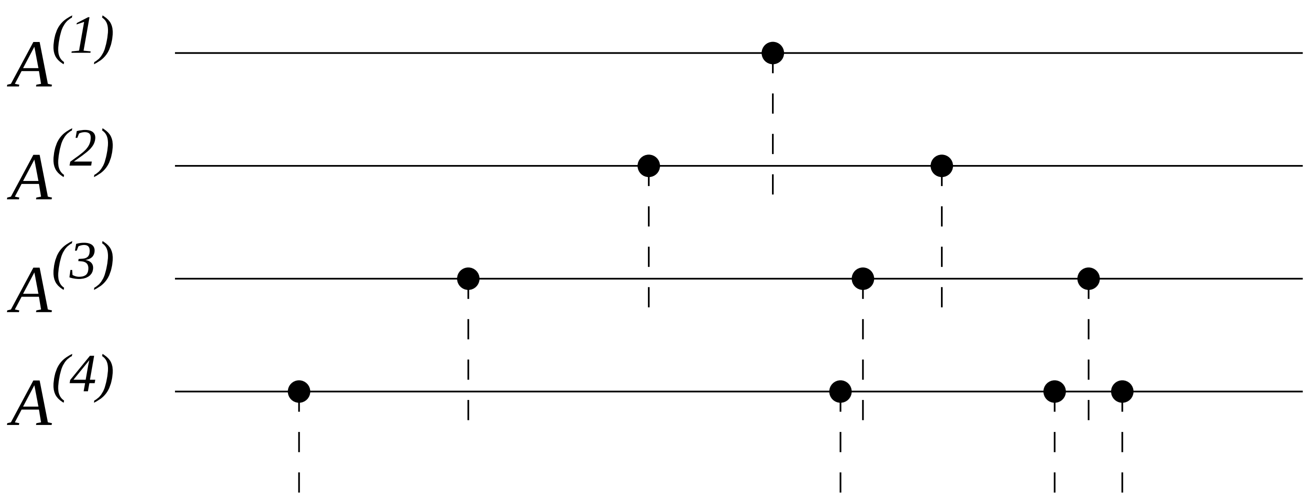
$$J = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \tan(2\theta) = \frac{2d}{b-a}$$

- Iteratively apply transformation to 2 rows and 2 columns of $A \in \mathbb{R}^{m\times m}$
- Loop over all pairs of rows/columns, quadratic convergence
- $O(m^2)$ steps, $O(m)$ operations per step $\Longrightarrow O(m^3)$ operation count

The Method of Bisection

- Idea: Search the real line for roots of $p(x) = \det(A - xI)$
- Finding roots from coefficients highly unstable, but $p(x)$ could be computed by elimination
- Important property: Eigenvalues of principal upper-left square submatrices $A^{(1)}, \ldots, A^{(m)}$ *interlace*

$$\lambda_j^{(k+1)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k+1)}$$



The Method of Bisection

- Because of the interlacing property: The number of negative eigenvalues of A equals the number of sign changes in the *Sturm sequence*

$$1,\ \det(A^{(1)}),\ \det(A^{(2)}), \ldots,\ \det(A^{(m)})$$

- Shift A to get number of eigenvalues in $(-\infty, b)$ and twice for $[a, b)$
- Three-term recurrence for the determinants:

$$\det(A^{(k)}) = a_k \det(A^{(k-1)}) - b_{k-1}^2 \det(A^{(k-2)})$$

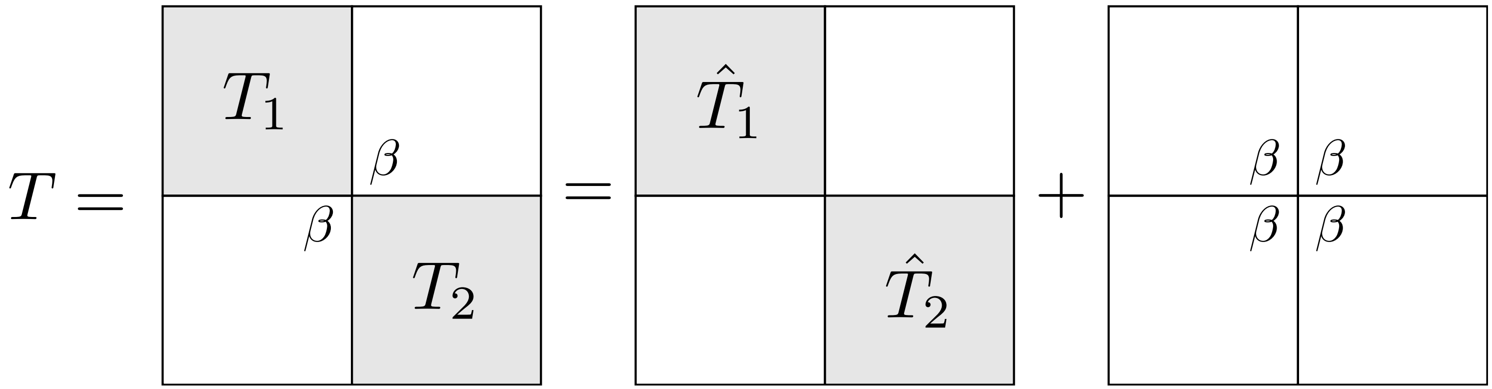
- With shift xI and $p^{(k)}(x) = \det(A^{(k)} - xI)$:

$$p^{(k)}(x) = (a_k - x)p^{(k-1)}(x) - b_{k-1}^2 p^{(k-2)}(x)$$

- $O(m \log(\epsilon_{\text{machine}})$ flops per eigenvalue, always high relative accuracy

The Divide-and-Conquer Algorithm

- Split symmetric tridiagonal T into submatrices:



- The sum of a 2×2 block-diagonal matrix and a rank-one correction
- Split T in equal sizes and compute eigenvalues of $\hat{T}_1, \hat{T}_2$ recursively
- Solve nonlinear problem to get eigenvalues of T from those of $\hat{T}_1, \hat{T}_2$

The Divide-and-Conquer Algorithm

- Suppose diagonalizations $\hat{T}_1 = Q_1 D_1 Q_1^T$ and $\hat{T}_2 = Q_2 D_2 Q_2^T$ have been computed. We then have

$$T = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \left(\begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} + \beta z z^T \right) \begin{bmatrix} Q_1^T & \\ & Q_2^T \end{bmatrix}$$

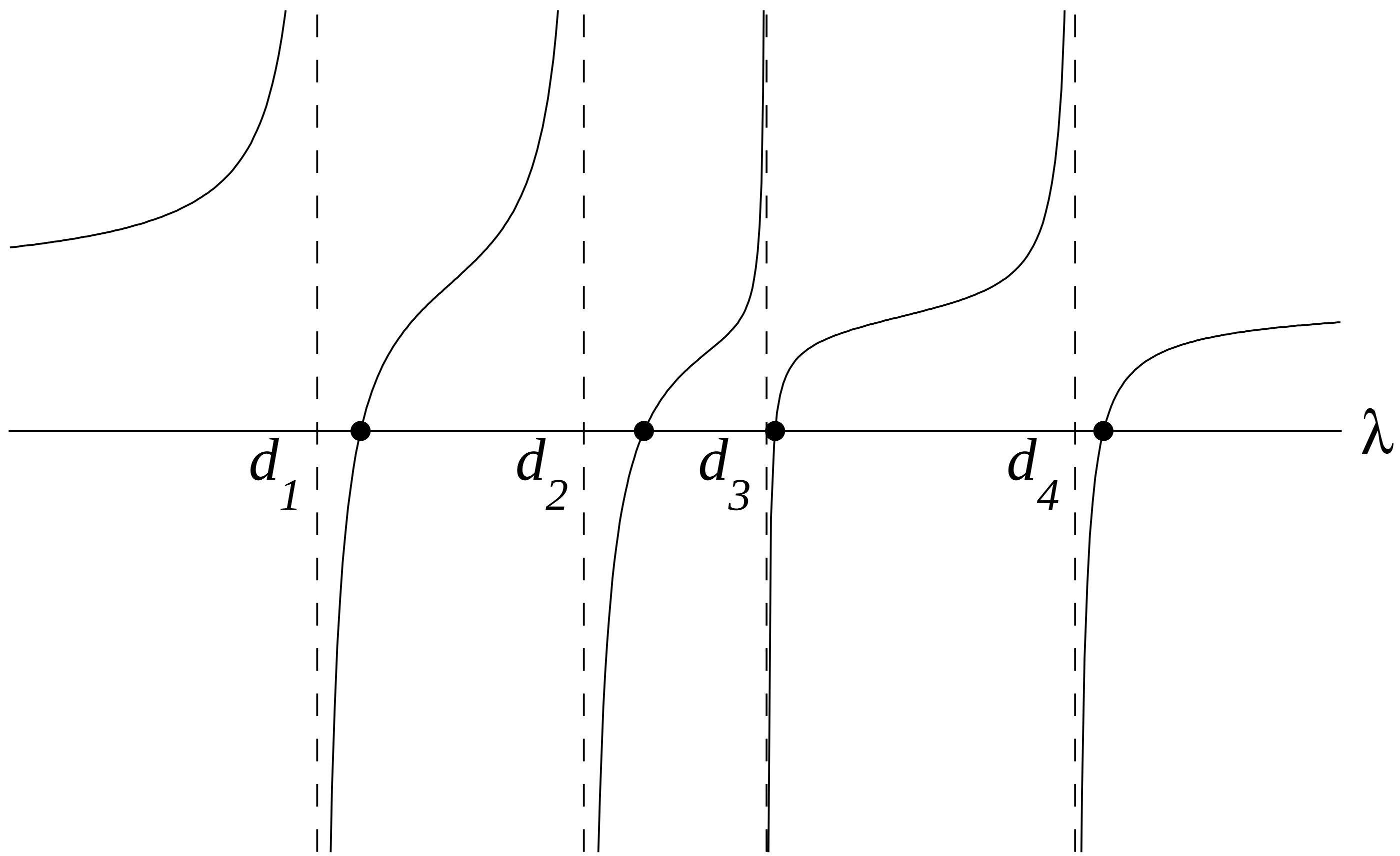
  with $z^T = (q_1^T, q_2^T)$, where q_1^T is last row of Q_1 and q_2^T is first row of Q_2

- This is a similarity transformation $\Longrightarrow$ Find eigenvalues of diagonal matrix plus rank-one correction

The Divide-and-Conquer Algorithm

- Eigenvalues of $D + ww^T$ are the roots of the rational function

$$f(\lambda) = 1 + \sum_{j=1}^{m} \frac{w_j^2}{d_j - \lambda}$$



The Divide-and-Conquer Algorithm

- Solve the *secular equation* $f(\lambda) = 0$ with nonlinear solver
- $O(m)$ flops per root, $O(m^2)$ flops for all roots
- Total cost for divide-and-conquer algorithm:

$$O\left(m^2 + 2\frac{m^2}{2^2} + 4\frac{m^2}{4^2} + 8\frac{m^2}{8^2} + \cdots + m\frac{m^2}{m^2}\right) = O(m^2)$$

- For computing eigenvalues only, most of the operations are spent in the tridiagonal reduction, and the constant in “Phase 2” is not important
- However, for computing eigenvectors, divide-and-conquer reduces Phase 2 to $4m^3/3$ flops compared to $6m^3$ for QR