

Lecture 14

Hessenberg/Tridiagonal Reduction

MIT 18.335J / 6.337J

Introduction to Numerical Methods

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Introducing Zeros by Similarity Transformations

- Try computing the Schur factorization $A = QTQ^*$ by applying Householder reflectors from left and right that introduce zeros:

$$\begin{array}{ccc} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} & \xrightarrow{Q_1^*} & \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} & \xrightarrow{Q_1} & \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \\ A & & Q_1^* A & & Q_1^* A Q_1 \end{array}$$

- The right multiplication destroys the zeros previously introduced
- We already knew this would not work, because of Abel's theorem
- However, the subdiagonal entries typically decrease in magnitude

The Hessenberg Form

- Instead, try computing an upper Hessenberg matrix H similar to A :

$$\begin{array}{ccc}
 \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} & \xrightarrow{Q_1^*} & \begin{bmatrix} \times & \times & \times & \times & \times \\ \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \end{bmatrix} & \xrightarrow{Q_1} & \begin{bmatrix} \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \end{bmatrix} \\
 A & & Q_1^* A & & Q_1^* A Q_1
 \end{array}$$

- This time the zeros we introduce are not destroyed
- Continue in a similar way with column 2:

$$\begin{array}{ccc}
 \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & \times & \times & \times & \times \end{bmatrix} & \xrightarrow{Q_1^*} & \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \end{bmatrix} & \xrightarrow{Q_1} & \begin{bmatrix} \times & \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ \times & \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \end{bmatrix} \\
 Q_1^* A Q_1 & & Q_2^* Q_1^* A Q_1 & & Q_2^* Q_1^* A Q_1 Q_2
 \end{array}$$

The Hessenberg Form

- After $m - 2$ steps, we obtain the Hessenberg form:

$$\underbrace{Q_{m-2}^* \cdots Q_2^* Q_1^*}_{Q^*} A \underbrace{Q_1 Q_2 \cdots Q_{m-2}}_Q = H = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

- For hermitian A , zeros are also introduced above diagonals

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \xrightarrow{Q_1^*} \begin{bmatrix} \times & \times & \times & \times & \times \\ \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ 0 & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} \times & \mathbf{\times} & 0 & 0 & 0 \\ \times & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & & \mathbf{\times} & \mathbf{\times} & \mathbf{\times} \\ & & & \mathbf{\times} & \mathbf{\times} \end{bmatrix}$$

A $Q_1^* A$ $Q_1^* A Q_1$

producing a tridiagonal matrix T after $m - 2$ steps

Householder Reduction to Hessenberg

Algorithm: Householder Hessenberg

for $k = 1$ **to** $m - 2$

$$x = A_{k+1:m,k}$$

$$v_k = \text{sign}(x_1) \|x\|_2 e_1 + x$$

$$v_k = v_k / \|v_k\|_2$$

$$A_{k+1:m,k:m} = A_{k+1:m,k:m} - 2v_k(v_k^* A_{k+1:m,k:m})$$

$$A_{1:m,k+1:m} = A_{1:m,k+1:m} - 2(A_{1:m,k+1:m} v_k) v_k^*$$

- Operation count (*not* twice Householder QR):

$$\sum_{k=1}^m 4(m-k)^2 + 4m(m-k) = \underbrace{4m^3/3 + 4m^3}_{QR} - 4m^3/2 = 10m^3/3$$

- For hermitian A , operation count is twice QR divided by two = $4m^3/3$

Stability of Householder Hessenberg

- The Householder Hessenberg reduction algorithm is backward stable:

$$\tilde{Q}\tilde{H}\tilde{Q}^* = A + \delta A, \quad \frac{\|\delta A\|}{\|A\|} = O(\epsilon_{\text{machine}})$$

where \tilde{Q} is an exactly unitary matrix based on \tilde{v}_k

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