18.335 Problem Set 1

Problem 1: Gaussian elimination

Trefethen, problem 20.4.

Problem 2: Asymptotic notation

This problem asks a few simple questions to make sure that you understand the asymptotic notations O, Ω, and Θ as defined in the handout in class, and also to make sure you are comfortable with simple proofs. (A detailed review of asymptotic notation can be found in any computer-science textbook, or on many sites online.)

(a) If $f(n)$ is $\Theta[F(n)]$ and $g(n)$ is $\Theta[G(n)]$ for nonnegative functions f, g, F, and G, prove that $f(n) + g(n)$ is $\Theta[F(n) + G(n)]$.

(b) Prove that $f(n)$ is $O[g(n)]$ if and only if $g(n)$ is $\Omega[f(n)]$. [For example, n^2 is $O(n^3)$ and n^3 is $\Omega(n^2)$.]

(c) If $f(n)$ is $O[F(n)]$, prove that any function that is $O[f(n) + c\,F(n)]$ must also be $O[F(n)]$ for any constant $c \neq 0$—that is, if we regard $O[\cdots]$ as a set of functions, prove $O[f(n)+c\,F(n)] \subseteq O[F(n)]$. [For example, $O(n^2+3n^3) = O(n^3)$.] Is it also true that $\Theta[f(n)+c\,F(n)] \subseteq \Theta[F(n)]$ for any $c \neq 0$ if $f(n)$ is $O[F(n)]$? Explain.

(d) Explain why the statement, "The running time of this algorithm is $O(n^2)$ or worse," cannot provide any information about the algorithm.

Problem 3: Caches and matrix multiplications

In class, we considered the performance and cache complexity of matrix multiplication $A = BC$, especially for square $m \times m$ matrices, and showed how to reduce the number of cache misses using various forms of blocking. In this problem, you will be comparing optimized matrix-matrix products to optimized matrix-vector products, using Matlab.

(a) The code `matmul_bycolumn.m` posted on the 18.335 web page computes $A = BC$ by multiplying B by each column of C individually (using Matlab's highly-optimized BLAS matrix-vector product). Benchmark this: plot the flop rate for square $m \times m$ matrices as a function of m, and also benchmark Matlab's built-in matrix-matrix product and plot it too. For example, Matlab code to benchmark Matlab's $m \times m$ products for $m = 1, \ldots, 1000$, storing the flop rate ($2m^3$/nanoseconds) in an array gflops(m), is:

```
gflops = zeros(1,1000);
for m = 1:1000
  A = rand(m,m);
  B = rand(m,m);
  t = 0;
  iters = 1;
  % run benchmark for at least 0.1 seconds
  while (t < 0.1)
    tic
    for iter = 1:iters
```

```
      C = A * B;
    end
    t = toc; % elapsed time in seconds
    iters = iters * 2;
  end
  gflops(m) = 2*m^3 * 1e-9 / (t * 2/iters);
  disp(sprintf('gflops for m=%d = %g after %d iters',m,gflops(m),iters/2));
  drawnow update;
end
plot([1:1000], gflops, 'r-')
```

(b) Compute the cache complexity (the asymptotic number of cache misses in the ideal-cache model, as in class) of an $m \times m$ matrix-vector product implemented the "obvious" way (a sequence of row·column dot products).

(c) Propose an algorithm for matrix-vector products that obtains a better asymptotic cache complexity (or at least a better constant coefficient, e.g. going from $\sim 3m^2$ to $\sim 2m^2$, even if it is still the same $\Theta[\cdots]$ complexity) by dividing the operation into some kind of blocks.

(d) Assuming Matlab uses something like your "improved" algorithm from part (c) to do matrix-vector products, compute the cache complexity of `matmul_bycolumn`. Compare this to the cache complexity of the blocked matrix-matrix multiply from class. Does this help to explain your results from part (a)?

Problem 4: Caches and backsubstitution

In this problem, you will consider the impact of caches (again in the ideal-cache model from class) on the problem of *backsubstitution*: solving $Rx = b$ for x, where R is an $m \times m$ upper-triangular matrix (such as might be obtained by Gaussian elimination). The simple algorithm you probably learned in previous linear-algebra classes (and reviewed in the book, lecture 17) is (processing the rows from bottom to top):

$x_m = b_m / r_{mm}$
for $j = m - 1$ **down to** 1
    $x_j = (b_j - \sum_{k=j+1}^{m} r_{jk} x_k) / r_{jj}$

Suppose that X and B are $m \times n$ matrices, and we want to solve $RX = B$ for X—this is equivalent to solving $Rx = b$ for n different right-hand sides b (the n columns of B). One way to solve the $RX = B$ for X is to apply the standard backsubstitution algorithm, above, to each of the n columns in sequence.

(a) Give the asymptotic cache complexity $Q(m, n; Z)$ (in asymptotic Θ notation, ignoring constant factors) of this algorithm for solving $RX = B$.

(b) Suppose $m = n$. Propose an algorithm for solving $RX = B$ that achieves a better asymptotic cache complexity (by cache-aware/blocking or cache-oblivious algorithms, your choice). Can you gain the factor of $1/\sqrt{Z}$ savings that we showed is possible for square-matrix multiplication?

18.335J / 6.337J Introduction to Numerical Methods
Fall 2010