15.081J/6.251J Introduction to Mathematical Programming

Lecture 6: The Simplex Method II

1 Outline

SLIDE 1

- Revised Simplex method
- The full tableau implementation
- Anticycling

2 Revised Simplex

SLIDE 2

Initial data: $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$

1. Start with basis $\boldsymbol{B} = [\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}]$ and $\boldsymbol{B}^{-1}$.
2. Compute $\boldsymbol{p}' = \boldsymbol{c}'_B \boldsymbol{B}^{-1}$
   $\overline{c}_j = c_j - \boldsymbol{p}' \boldsymbol{A}_j$
   - If $\overline{c}_j \geq 0$; x optimal; stop.
   - Else select $j : \overline{c}_j < 0$.

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3. Compute $\boldsymbol{u} = \boldsymbol{B}^{-1} \boldsymbol{A}_j$.
   - If $\boldsymbol{u} \leq \boldsymbol{0} \Rightarrow$ cost unbounded; stop
   - Else
4. $\theta^* = \min\limits_{1 \leq i \leq m, u_i > 0} \dfrac{x_{B(i)}}{u_i} = \dfrac{u_{B(l)}}{u_l}$
5. Form a new basis $\overline{B}$ by replacing $\boldsymbol{A}_{B(l)}$ with $\boldsymbol{A}_j$.
6. $y_j = \theta^*$, $y_{B(i)} = x_{B(i)} - \theta^* u_i$

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7. Form $[\boldsymbol{B}^{-1} | \boldsymbol{u}]$
8. Add to each one of its rows a multiple of the lth row in order to make the last column equal to the unit vector $\boldsymbol{e}_l$.
   The first m columns is $\overline{\boldsymbol{B}}^{-1}$.

2.1 Example

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$$
\begin{array}{llllll}
\min & x_1+ & 5x_2 & -2x_3 & \\
\text{s.t.} & x_1+ & x_2+ & x_3 & \leq 4 \\
 & x_1 & & & \leq 2 \\
 & & & x_3 & \leq 3 \\
 & & 3x_2+ & x_3 & \leq 6 \\
 & x_1, & x_2, & x_3 & \geq 0
\end{array}
$$

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B $= \{\boldsymbol{A}_1, \boldsymbol{A}_3, \boldsymbol{A}_6, \boldsymbol{A}_7\}$, BFS: $\boldsymbol{x} = (2,0,2,0,0,1,4)'$
$\overline{\boldsymbol{c}}' = (0,7,0,2,-3,0,0)$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$(u_1, u_3, u_6, u_7)' = \boldsymbol{B}^{-1}\boldsymbol{A}_5 = (1,-1,1,1)'$
$\theta^* = \min\left(\frac{2}{1}, \frac{1}{1}, \frac{4}{1}\right) = 1, \quad l = 6$
$l = 6$ ($\boldsymbol{A}_6$ exits the basis). SLIDE 7

$$[\boldsymbol{B}^{-1}|\boldsymbol{u}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \overline{\boldsymbol{B}}^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

2.2 Practical issues

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- **Numerical Stability**
  $\boldsymbol{B}^{-1}$ needs to be computed from scratch once in a while, as errors accumulate
- **Sparsity**
  $\boldsymbol{B}^{-1}$ is represented in terms of sparse triangular matrices

3 Full tableau implementation

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$-\boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{b}$	$\boldsymbol{c}' - \boldsymbol{c}'_B\boldsymbol{B}^{-1}\boldsymbol{A}$
$\boldsymbol{B}^{-1}\boldsymbol{b}$	$\boldsymbol{B}^{-1}\boldsymbol{A}$

or, in more detail,

$-\boldsymbol{c}'_B\boldsymbol{x}_B$	$\overline{c}_1$	$\dots$	$\overline{c}_n$
$x_{B(1)}$	|		|
$\vdots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_1$	$\dots$	$\boldsymbol{B}^{-1}\boldsymbol{A}_n$
$x_{B(m)}$	|		|

3.1 Example

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$$\begin{array}{lrcrcrcl}
\min & -10x_1 & - & 12x_2 & - & 12x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & \leq & 20 \\
& 2x_1 & + & x_2 & + & 2x_3 & \leq & 20 \\
& 2x_1 & + & 2x_2 & + & x_3 & \leq & 20 \\
& \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0}
\end{array}$$

$$\begin{array}{lrcrcrcrcrcrcl}
\min & -10x_1 & - & 12x_2 & - & 12x_3 & & & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & & & = & 20 \\
& 2x_1 & + & x_2 & + & 2x_3 & & & + & x_5 & & & = & 20 \\
& 2x_1 & + & 2x_2 & + & x_3 & & & & & + & x_6 & = & 20 \\
& \multicolumn{13}{l}{x_1, \ldots, x_6 \geq 0}
\end{array}$$

BFS: $\boldsymbol{x} = (0, 0, 0, 20, 20, 20)'$
B=$[\boldsymbol{A}_4, \boldsymbol{A}_5, \boldsymbol{A}_6]$

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		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2*	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

$\overline{\boldsymbol{c}}' = \boldsymbol{c}' - \boldsymbol{c}_B'\boldsymbol{B}^{-1}\boldsymbol{A} = \boldsymbol{c}' = (-10, -12, -12, 0, 0, 0)$

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		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1*	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

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		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5*	0	1	-1.5	1

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		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

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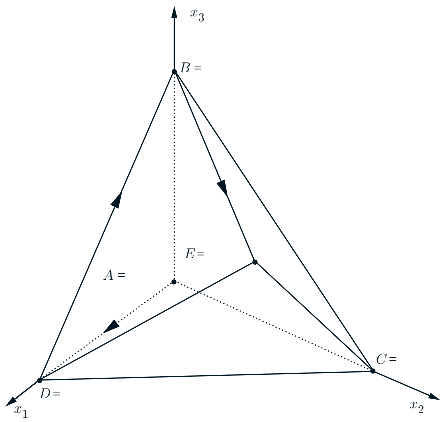

4 Comparison of implementations

SLIDE 16

	Full tableau	**Revised simplex**
Memory	$O(mn)$	$O(m^2)$
Worst-case time	$O(mn)$	$O(mn)$
Best-case time	$O(mn)$	$O(m^2)$

5 Anticycling

5.1 Degeneracy in Practice

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Does degeneracy really happen in practice?

$$\sum_{j=1}^{n} x_{ij} = 1$$
$$\sum_{i=1}^{n} x_{ij} = 1$$
$$x_{ij} \geq 0$$

$n!$ vertices:
For each vertex $\exists\ 2^{n-1}n^{n-2}$ different bases ($n = 8$) for each vertex $\exists\ 33,554,432$ bases.

5.2 Perturbations

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$$(P)\ \min\ \boldsymbol{c}'\boldsymbol{x} \qquad\qquad (P_\epsilon)\ \min\ \boldsymbol{c}'\boldsymbol{x}$$

$$\text{s.t.}\quad \boldsymbol{Ax} = \boldsymbol{b} \qquad\qquad \text{s.t.}\quad \boldsymbol{Ax} = \boldsymbol{b} + \begin{pmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^m \end{pmatrix}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \qquad\qquad \boldsymbol{x} \geq \boldsymbol{0}.$$

5.2.1 Theorem

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$\exists\ \epsilon_1 > 0$: for all $0 < \epsilon < \epsilon_1$

$$\boldsymbol{Ax} = \boldsymbol{b} + \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon^m \end{pmatrix}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

is non-degenerate.

5.2.2 Proof

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Let $\boldsymbol{B}_1, \ldots, \boldsymbol{B}_r$ be all the bases.

$$\boldsymbol{B}_r^{-1}\left[\boldsymbol{b} + \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon^m \end{pmatrix}\right] = \begin{bmatrix} \overline{b}_1^r + \boldsymbol{B}_{11}^r\epsilon + \cdots + \boldsymbol{B}_{1m}^r\epsilon^m \\ \vdots \\ \overline{b}_m^r + \boldsymbol{B}_{m1}^r\epsilon + \cdots + \boldsymbol{B}_{mm}^r\epsilon^m \end{bmatrix}$$

where:

$$\boldsymbol{B}_r^{-1} = \begin{bmatrix} \boldsymbol{B}_{11}^r & \cdots & \boldsymbol{B}_{1m}^r \\ \vdots & & \vdots \\ \boldsymbol{B}_{m1}^r & \cdots & \boldsymbol{B}_{mm}^r \end{bmatrix}, \boldsymbol{B}_r^{-1}\boldsymbol{b} = \begin{bmatrix} \overline{b}_1^r \\ \vdots \\ \overline{b}_m^r \end{bmatrix}$$

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- $\overline{b}_i^r + \boldsymbol{B}_{i1}^r\theta + \cdots + \boldsymbol{B}_{im}^r\theta^m$ is a polynomial in θ
- Roots $\theta_{i,1}^r, \theta_{i,2}^r, \ldots, \theta_{i,m}^r$
- If $\epsilon \neq \theta_{i,1}^r, \ldots, \theta_{i,m}^r \Rightarrow \overline{b}_i^r + \boldsymbol{B}_{i1}^r\epsilon + \cdots + \boldsymbol{B}_{im}^r\epsilon^m \neq 0$.
- Let ϵ_1 the smallest positive root $\Rightarrow 0 < \epsilon < \epsilon_1$ all RHS are $\neq 0 \Rightarrow$ non-degeneracy.

5.3 Lexicography

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- $\boldsymbol{u}$ is **lexicographically larger** than $\boldsymbol{v}$, $\boldsymbol{u} \stackrel{L}{>} \boldsymbol{v}$, if $\boldsymbol{u} \neq \boldsymbol{v}$ and the first nonzero component of $\boldsymbol{u} - \boldsymbol{v}$ is positive.
- Example:

$$(0,\ 2,\ 3,\ 0) \stackrel{L}{>} (0,\ 2,\ 1,\ 4),$$
$$(0,\ 4,\ 5,\ 0) \stackrel{L}{<} (1,\ 2,\ 1,\ 2).$$

5.4 Lexicography-Pertubation

5.4.1 Theorem

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Let $\boldsymbol{B}$ be a basis of $\boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$. Then $\boldsymbol{B}$ is feasible for $\boldsymbol{Ax} = \boldsymbol{b} + (\epsilon, \ldots, \epsilon^m)', \quad \boldsymbol{x} \geq \boldsymbol{0}$ for sufficiently small ϵ if and only if

$$\boldsymbol{u}_i = (\overline{b}_i, B_{i1}, \ldots, B_{im}) \stackrel{L}{>} \boldsymbol{0}, \forall\, i$$

$$\boldsymbol{B}^{-1} = (B_{ij})$$
$$(\boldsymbol{B}^{-1}\boldsymbol{b})_i = (\overline{b}_i)$$

5.4.2 Proof

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$\boldsymbol{B}$ is feasible for peturbed problem "$\Leftrightarrow$" $\boldsymbol{B}^{-1}\left(\boldsymbol{b} + (\epsilon, \ldots, \epsilon^m)'\right) \geq \boldsymbol{0} \Leftrightarrow$
$\overline{b}_i + \boldsymbol{B}_{i1}\epsilon + \cdots + \boldsymbol{B}_{im}\epsilon^m \geq 0\ \forall\, i$
$\Leftrightarrow$ First non-zero component of $\boldsymbol{u}_i = (\overline{b}_i, B_{i1}, \ldots, B_{im})$ is positive $\forall\, i$.

5.5 Summary

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1. We start with: $(P):\ \boldsymbol{Ax} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}$
2. We introduce (P_ϵ): $\boldsymbol{Ax} = \boldsymbol{b} + (\epsilon, \ldots, \epsilon^m)',\ \boldsymbol{x} \geq \boldsymbol{0}$
3. A basis is feasible + non-degenerate in $(P_\epsilon) \Leftrightarrow \boldsymbol{u}_i \stackrel{L}{>} \boldsymbol{0}$ in (P).
4. If we maintain $\boldsymbol{u}_i \stackrel{L}{>} \boldsymbol{0}$ in $(P) \Rightarrow (P_\epsilon)$ is non-degenerate $\Rightarrow$ Simplex is finite in (P_ϵ) for sufficiently small ϵ.

5.6 Lexicographic pivoting rule

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1. Choose an entering column $\boldsymbol{A}_j$ arbitrarily, as long as $\overline{c}_j < 0$; $\boldsymbol{u} = \boldsymbol{B}^{-1}\boldsymbol{A}_j$.
2. For each i with $u_i > 0$, divide the ith row of the tableau (including the entry in the zeroth column) by u_i and choose the lexicographically smallest row. If row l is lexicographically smallest, then the lth basic variable $x_{B(l)}$ exits the basis.

5.6.1 Example

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- $j = 3$

1	0	5	3	$\cdots$
2	4	6	-1	$\cdots$
3	0	7	9	$\cdots$

- $x_{B(1)}/u_1 = 1/3$ and $x_{B(3)}/u_3 = 3/9 = 1/3$.
- We divide the first and third rows of the tableau by $u_1 = 3$ and $u_3 = 9$, respectively, to obtain:

1/3	0	5/3	1	$\cdots$
*	*	*	*	$\cdots$
1/3	0	7/9	1	$\cdots$

- Since $7/9 < 5/3$, the third row is chosen to be the pivot row, and the variable $x_{B(3)}$ exits the basis.

5.6.2 Uniqueness

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- Why lexicographic pivoting rule always leads to a unique choice for the exiting variable?
- Otherwise, two rows in tableau proportional $\Rightarrow$ rank$(\boldsymbol{B}^{-1}\boldsymbol{A}) < m \Rightarrow$ rank$(\boldsymbol{A}) < m$

5.7 Theorem

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If simplex starts with all the rows in the simplex tableau, other than the zeroth row, lexicographically positive and the lexicographic pivoting rule is followed, then

(a) Every row of the simplex tableau, other than the zeroth row, remains lexicographically positive throughout the algorithm.

(b) The zeroth row strictly increases lexicographically at each iteration.

(c) The simplex method terminates after a finite number of iterations.

5.8 Smallest subscript pivoting rule

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1. Find the smallest j for which the reduced cost $\overline{c}_j$ is negative and have the column $\boldsymbol{A}_j$ enter the basis.
2. Out of all variables x_i that are tied in the test for choosing an exiting variable, select the one with the smallest value of i.

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