

## Circuit Analysis using the Node and Mesh Methods

We have seen that using Kirchhoff's laws and Ohm's law we can analyze any circuit to determine the operating conditions (the currents and voltages). The challenge of formal circuit analysis is to derive the smallest set of simultaneous equations that completely define the operating characteristics of a circuit.

In this lecture we will develop two very powerful methods for analyzing any circuit: The **node method** and the **mesh method**. These methods are based on the systematic application of Kirchhoff's laws. We will explain the steps required to obtain the solution by considering the circuit example shown on Figure 1.

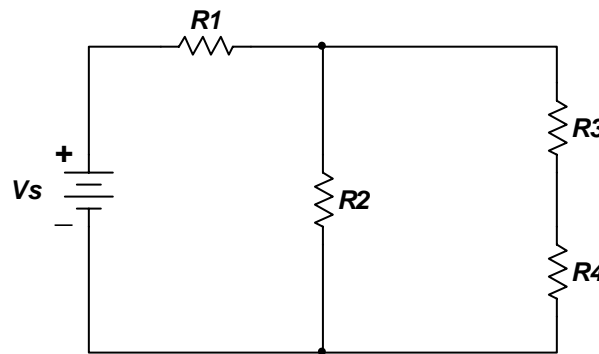


Figure 1. A typical resistive circuit.

The Node Method.

A voltage is always defined as the potential difference between two points. When we talk about the voltage at a certain point of a circuit we imply that the measurement is performed between that point and some other point in the circuit. In most cases that other point is referred to as ground.

The node method or the node voltage method, is a very powerful approach for circuit analysis and it is based on the application of KCL, KVL and Ohm's law. The procedure for analyzing a circuit with the node method is based on the following steps.

1. Clearly label all circuit parameters and distinguish the unknown parameters from the known.
2. Identify all nodes of the circuit.
3. Select a node as the reference node also called the ground and assign to it a potential of 0 Volts. All other voltages in the circuit are measured with respect to the reference node.
4. Label the voltages at all other nodes.
5. Assign and label polarities.
6. Apply KCL at each node and express the branch currents in terms of the node voltages.
7. Solve the resulting simultaneous equations for the node voltages.

8. Now that the node voltages are known, the branch currents may be obtained from Ohm's law.

We will use the circuit of Figure 1 for a step by step demonstration of the node method.

Figure 2 shows the implementation of steps 1 and 2. We have labeled all elements and identified all relevant nodes in the circuit.

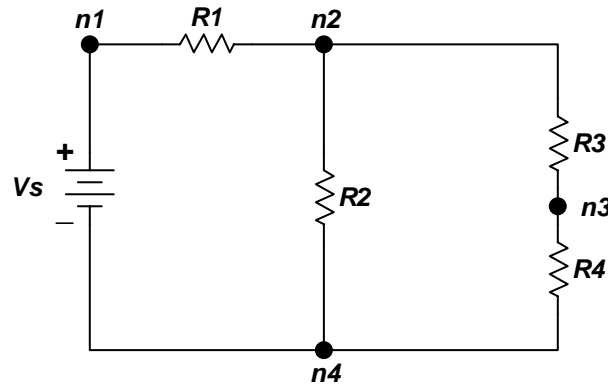


Figure 2. Circuit with labeled nodes.

The third step is to select one of the identified nodes as the reference node. We have four different choices for the assignment. In principle any of these nodes may be selected as the reference node. However, some nodes are more useful than others. Useful nodes are the ones which make the problem easier to understand and solve. There are a few general guidelines that we need to remember as we make the selection of the reference node.

1. A useful reference node is one which has the largest number of elements connected to it.
2. A useful reference node is one which is connected to the maximum number of voltage sources.

For our example circuit the selection of node  $n4$  as the reference node is the best choice. (equivalently we could have selected node  $n1$  as our reference node.)

The next step is to label the voltages at the selected nodes. Figure 3 shows the circuit with the labeled nodal voltages. The reference node is assigned voltage 0 Volts indicated by the ground symbol. The remaining node voltages are labeled  $v1$ ,  $v2$ ,  $v3$ .

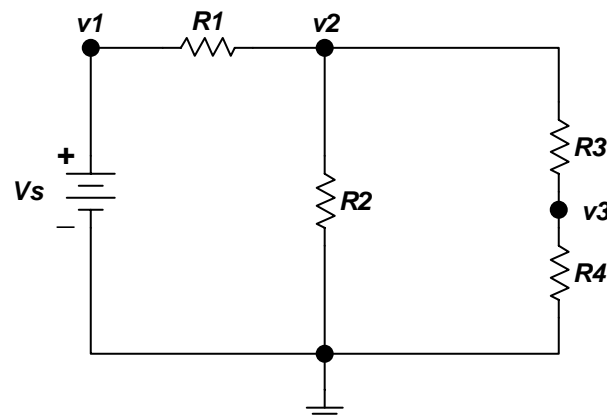


Figure 3. Circuit with assigned nodal voltages.

For the next step we assign current flow and polarities, see Figure 4.

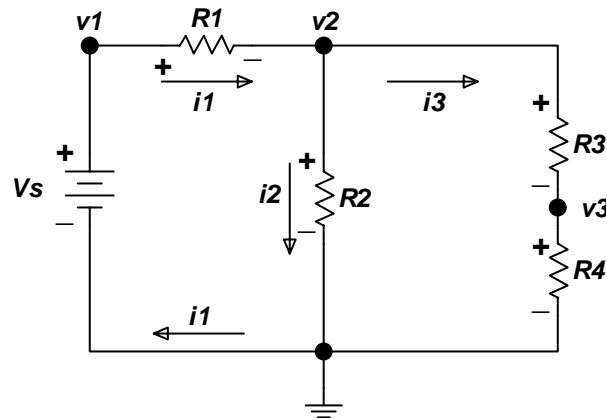


Figure 4. Example circuit with assigned node voltages and polarities.

Before proceeding let's look at the circuit shown on Figure 4 bit closer. Note that the problem is completely defined. Once we determine the values for the node voltages  $v_1$ ,  $v_2$ ,  $v_3$  we will be able to completely characterize this circuit. So let's go on to calculate the node voltages by applying KCL at the designated nodes.

For node  $n_1$  since the voltage of the voltage source is known we may directly label the voltage  $v_1$  as

$$v_1 = V_s \quad (4.1)$$

and as a result we have reduced the number of unknowns from 3 to 2.

KCL at node  $n_2$  associated with voltage  $v_2$  gives:

$$i_1 = i_2 + i_3 \quad (4.2)$$

The currents  $i_1$ ,  $i_2$ ,  $i_3$  are expressed in terms of the voltages  $v_1$ ,  $v_2$ ,  $v_3$  as follows.

$$i_1 = \frac{V_s - v_2}{R_1} \quad (4.3)$$

$$i_2 = \frac{v_2}{R_2} \quad (4.4)$$

$$i_3 = \frac{v_2 - v_3}{R_3} \quad (4.5)$$

By combining Eqs. 4.2 – 4.5 we obtain

$$\frac{V_s - v_2}{R_1} - \frac{v_2}{R_2} - \frac{v_2 - v_3}{R_3} = 0 \quad (4.6)$$

Rewrite the above expression as a linear function of the unknown voltages **v2 and v3** gives.

$$v_2 \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) - v_3 \frac{1}{R_3} = V_s \frac{1}{R_1} \quad (4.7)$$

KCL at node **n3** associated with voltage **v3** gives:

$$\frac{v_2 - v_3}{R_3} - \frac{v_3}{R_4} = 0 \quad (4.8)$$

or

$$-v_2 \frac{1}{R_3} + v_3 \left( \frac{1}{R_3} + \frac{1}{R_4} \right) = 0 \quad (4.9)$$

The next step is to solve the simultaneous equations 4.7 and 4.9 for the node voltages **v2** and **v3**.

Although it is easy to solve Eqs. (4.8) and (4.9) directly it is useful to rewrite them in matrix form as follows.

$$\begin{aligned} v_2 \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) - v_3 \frac{1}{R_3} &= V_s \frac{1}{R_1} \\ -v_2 \left( \frac{1}{R_3} \right) + v_3 \left( \frac{1}{R_3} + \frac{1}{R_4} \right) &= 0 \end{aligned} \quad (4.10)$$

Or

$$\begin{bmatrix} 1 + \frac{R_1}{R_2} + \frac{R_1}{R_3} & -\frac{R_1}{R_3} \\ -\frac{R_1}{R_3} & \frac{R_1}{R_3} + \frac{R_1}{R_4} \end{bmatrix} \cdot \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix} \quad (4.11)$$

or equivalently.

$$\overline{\mathbf{A}} \cdot \overline{\mathbf{v}} = \overline{\mathbf{V}} \quad (4.12)$$

In defining the set of simultaneous equations we want to end up with a simple and consistent form. The simple rules to follow and check are:

- Place all sources (current and voltage) on the right hand side of the equation, as inhomogeneous drive terms,
- The terms comprising each element on the diagonal of matrix  $\overline{\mathbf{A}}$  must have the same sign. For example, there is no combination  $\frac{R1}{R2} - \frac{R1}{R3}$ . If an element on the diagonal is comprised of both positive and negative terms there must be a sign error somewhere.
- If you arrange so that all diagonal elements are positive, then the off-diagonal elements are negative and the matrix is symmetric:  $A_{ij} = A_{ji}$ . If the matrix does not have this property there is a mistake somewhere.

Putting the circuit equations in the above form guarantees that there is a solution consisting of a real set of currents.

Once we put the equations in matrix form and perform the checks detailed above the solutions then there is a solution if the  $\det \overline{\mathbf{A}} \neq 0$ . The unknown voltage  $\mathbf{v}_k$  are given by:

$$\mathbf{v}_k = \frac{\det \overline{\mathbf{A}}_k}{\det \overline{\mathbf{A}}} \quad (4.13)$$

Where  $\overline{\mathbf{A}}_k$  is the matrix  $\overline{\mathbf{A}}$  with the kth column replaced by the vector  $\mathbf{V}$ .

For our example the voltages  $\mathbf{v}_2$  and  $\mathbf{v}_3$  are given by:

$$\mathbf{v}_2 = \frac{R_2(R_3 + R_4)V_s}{R_1R_2 + R_1R_3 + R_2R_3 + R_1R_4 + R_2R_4} \quad (4.14)$$

$$\mathbf{v}_3 = \frac{R_2R_4 V_s}{R_1R_2 + R_1R_3 + R_2R_3 + R_1R_4 + R_2R_4} \quad (4.15)$$

We can express the above results compactly by introducing the quantity

$$R_{eff} = \frac{R_2(R_3 + R_4)}{R_2 + R_3 + R_4} \quad (4.16)$$

This resistance **Reff** arises naturally in the problem as you can see by redrawing the circuit as shown on Figure 5.

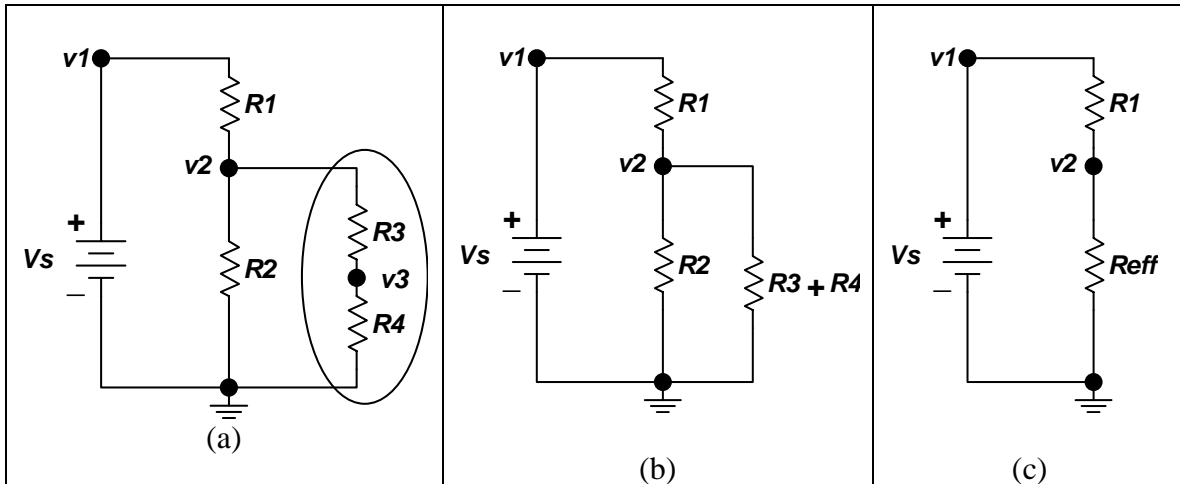


Figure 5. Circuit simplification

In terms of  $R_{eff}$  the solutions become:

$$v_2 = V_s \frac{R_{eff}}{R_1 + R_{eff}} \quad (4.17)$$

$$v_3 = v_2 \frac{R_4}{R_3 + R_4} \quad (4.18)$$

The result for  $v_3$  becomes clear if we consider the part of the circuit enclosed by the ellipse on Figure 5(a)

Given the voltages at these nodes, we can then use Ohm's law to calculate the currents.

$$i_1 = \frac{V_s}{R_1 + R_{eff}} \quad (4.19)$$

$$i_2 = \frac{v_2}{R_2} \quad (4.20)$$

and

$$i_3 = \frac{v_3}{R_3 + R_4} \quad (4.21)$$

So, the node voltage method provides an algorithm for calculating the voltages at the nodes of a circuit. Provided one can specify the connectivity of elements between nodes, then one can write down a set of simultaneous equations for the voltages at the nodes. Once these voltages have been solved for, then the currents are calculated via Ohm's law.

### Nodal analysis with floating voltage sources. The Supernode.

If a voltage source is not connected to the reference node it is called a floating voltage source and special care must be taken when performing the analysis of the circuit.

In the circuit of Figure 6 the voltage source  $V2$  is not connected to the reference node and thus it is a floating voltage source.

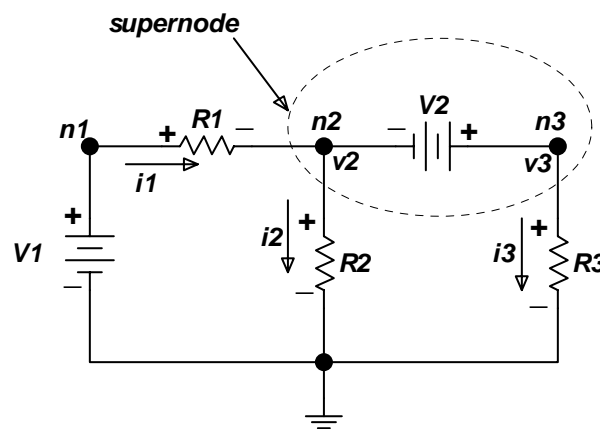


Figure 6. Circuit with a supernode.

The part of the circuit enclosed by the dotted ellipse is called a supernode. Kirchhoff's current law may be applied to a supernode in the same way that it is applied to any other regular node. This is not surprising considering that KCL describes charge conservation which holds in the case of the supernode as it does in the case of a regular node.

In our example application of KCL at the supernode gives

$$i1 = i2 + i3 \quad (4.22)$$

In term of the node voltages Equation (4.22) becomes:

$$\frac{V1 - v2}{R1} = \frac{v2}{R2} + \frac{v3}{R3} \quad (4.23)$$

The relationship between node voltages  $v1$  and  $v2$  is the constraint that is needed in order to completely define the problem. The constraint is provided by the voltage source  $V2$ .

$$V2 = v3 - v2 \quad (4.24)$$

Combining Equations (4.23) and (4.24) gives

$$v_2 = \frac{\frac{V_1}{R_1} - \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \quad (4.25)$$

$$v_3 = \frac{\frac{V_1}{R_1} - \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} - V_2 \quad (4.26)$$

Having determined the node voltages, the calculation of the branch currents follows from a simple application of Ohm's law.

#### Example 4.1 Nodal analysis with a supernode

The circuit in Figure 7 contains two voltage sources and with our assignment of the reference node voltage source  $V_2$  is a floating voltage source. As indicated in the figure, the supernode now encloses the voltage source as well as the resistor element  $R_4$  which is parallel with it.

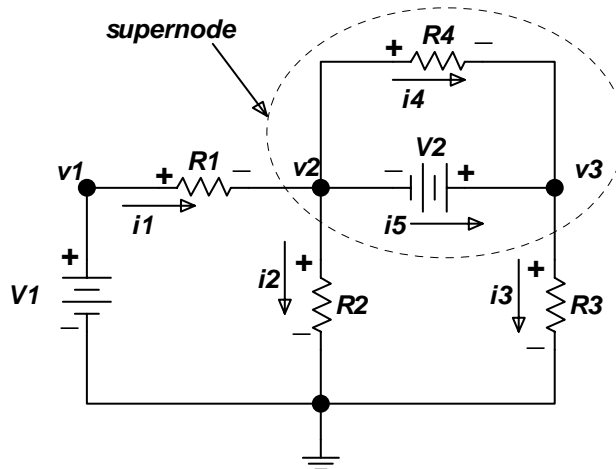


Figure 7. Another supernode example

First we notice that the current  $i_4$  through resistor  $R_4$  is given by

$$i_4 = -\frac{V_2}{R_4} \quad (4.27)$$

Where the negative sign denotes that the current direction is opposite to the one indicated.



Applying KCL at the supernode we have

$$i_1 = i_2 + i_3 \Rightarrow \frac{V_1 - v_2}{R_1} = \frac{v_2}{R_2} + \frac{v_3}{R_3} \quad (4.28)$$

The floating voltage source provides a constraint between  $v_2$  and  $v_3$ , such that  $V_2 = v_3 - v_2$ , and thus Equation (4.28) becomes

$$v_2 = \frac{\frac{V_1}{R_1} - \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \quad (4.29)$$

And the node voltage  $v_3$  follows.

$$v_3 = \frac{\frac{V_1}{R_1} - \frac{V_2}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} + V_2 \quad (4.30)$$

#### Example 4.1 Nodal analysis with current sources

Determine the node voltages  $v_1$ ,  $v_2$ , and  $v_3$  of the circuit in Figure 8.

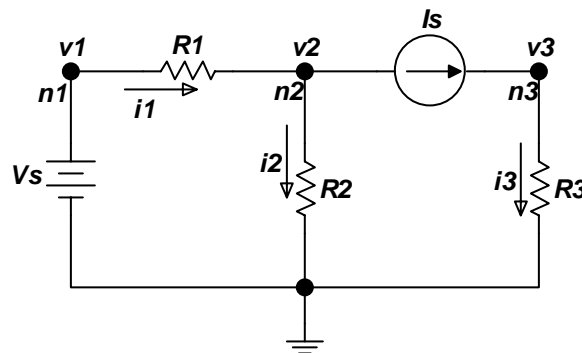


Figure 8. Circuit with voltage and current source.

We have applied the first five steps of the nodal method and now we are ready to apply KCL to the designated nodes. In this example, the current source  $I_s$  constrains the current  $i_3$  such that  $i_3 = I_s$ .

KCL at node  $n_2$  gives

$$i_1 = i_2 + I_s \quad (4.31)$$

And with the application of Ohm's law

$$\frac{V_s - v_2}{R_1} = \frac{v_2}{R_2} + \frac{v_3}{R_3} \quad (4.32)$$

Where we have used  $v_1 = V_s$  at node  $n_1$ .

The current source provides a constraint for the voltage  $v_3$  at node  $n_3$ .

$$v_3 = I_s R_3 \quad (4.33)$$

Combining Equations (4.32) and (4.33) we obtain the unknown node voltage  $v_2$

$$v_2 = \frac{\frac{V_s}{R_1} - I_s R_3}{\frac{1}{R_1} + \frac{1}{R_2}} \quad (4.34)$$

## The Mesh Method

The mesh method uses the mesh currents as the circuit variables. The procedure for obtaining the solution is similar to that followed in the Node method and the various steps are given below.

1. Clearly label all circuit parameters and distinguish the unknown parameters from the known.
2. Identify all meshes of the circuit.
3. Assign mesh currents and label polarities.
4. Apply KVL at each mesh and express the voltages in terms of the mesh currents.
5. Solve the resulting simultaneous equations for the mesh currents.
6. Now that the mesh currents are known, the voltages may be obtained from Ohm's law.

A mesh is defined as a loop which does not contain any other loops. Our circuit example has three loops but only two meshes as shown on Figure 9. Note that we have assigned a ground potential to a certain part of the circuit. Since the definition of ground potential is fundamental in understanding circuits this is a good practice and thus will continue to designate a reference (ground) potential as we continue to design and analyze circuits regardless of the method used in the analysis.

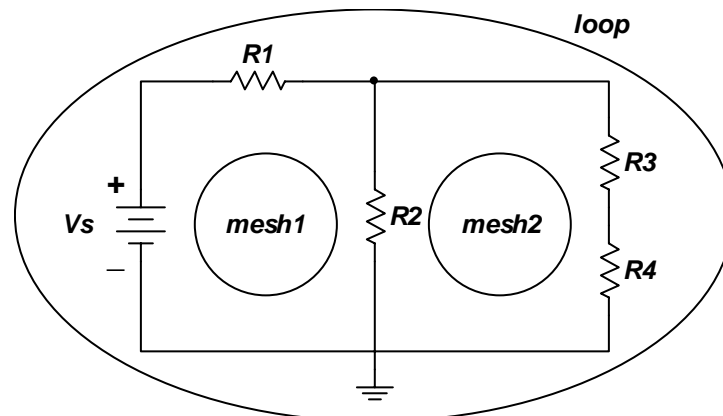


Figure 9. Identification of the meshes

The meshes of interest are mesh1 and mesh2.

For the next step we will assign mesh currents, define current direction and voltage polarities.

The direction of the mesh currents  $I_1$  and  $I_2$  is defined in the clockwise direction as shown on Figure 10. This definition for the current direction is arbitrary but it helps if we maintain consistence in the way we define these current directions. Note that in certain parts of the mesh the branch current may be the same as the current in the mesh. The branch of the circuit containing resistor  $R_2$  is shared by the two meshes and thus the branch current (the current flowing through  $R_2$ ) is the difference of the two mesh currents. (Note that in order to distinguish between the mesh currents and the branch

currents by using the symbol  $I$  for the mesh currents and the symbol  $i$  for the branch currents.)

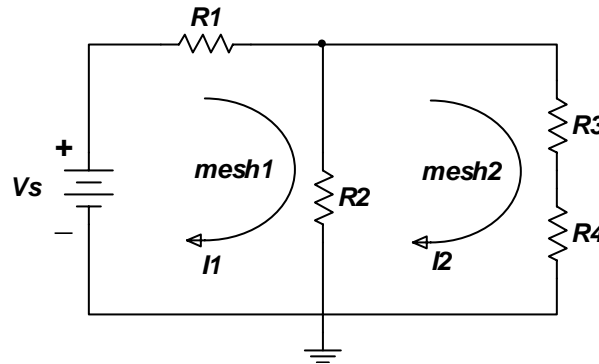


Figure 10. labeling mesh current direction

Now let's turn our attention in labeling the voltages across the various branch elements. We choose to assign the voltage labels to be consistent with the direction of the indicated mesh currents. In the case where a certain branch is shared by two meshes as is the case in our example with the branch that contains resistor  $R2$  the labeling of the voltage is done for each mesh consistent with the assigned direction of the mesh current.

In this, our first encounter with mesh analysis let's consider the each mesh separately and apply KVL around the loop following the defined direction of the mesh current.

Considering mesh1.

For clarity we have separated mesh1 from the circuit on Figure 11. In doing this, care must be taken to carry all the information of the shared branches. Here we indicate the direction of mesh current  $I2$  on the shared branch.

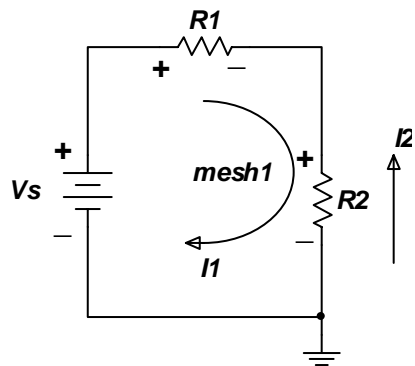


Figure 11. Sub-circuit for mesh1

Apply KVL to mesh1.

Starting at the upper left corner and proceeding in a clock-wise direction the sum of voltages across all elements encountered is:

$$I_1 R_1 + (I_1 - I_2) R_2 - V_s = 0 \quad (4.35)$$

Similarly, consideration of mesh2 is shown on Figure 12. Note again that we have indicated the direction of the mesh current  $I_1$  on the shared circuit branch.

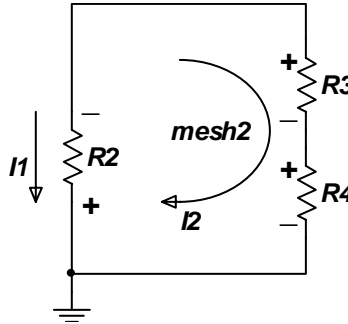


Figure 12. Sub-circuit for mesh2

Apply KVL to mesh2

Starting at the upper right corner and proceeding in a clock-wise direction the sum of voltages across all elements encountered is:

$$I_2 (R_3 + R_4) + (I_2 - I_1) R_2 = 0 \quad (4.36)$$

Keeping in mind that the unknowns of the problem are the mesh currents  $I_1$  and  $I_2$  we rewrite the mesh equations (4.35) and (4.36) as

$$I_1 (R_1 + R_2) - I_2 R_2 = V_s \quad (4.37)$$

$$-I_1 R_2 + I_2 (R_2 + R_3 + R_4) = 0 \quad (4.38)$$

In matrix form equations (4.37) and (4.38) become,

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 + R_4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix} \quad (4.39)$$

Equation (4.39) may now be solved for the mesh currents  $I_1$  and  $I_2$ .

It is evident from Figure 13 that the branch currents are:

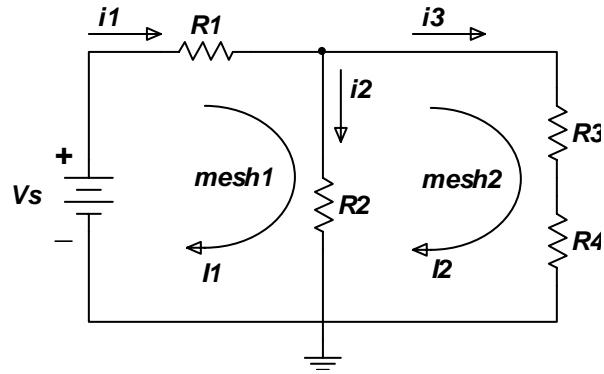


Figure 13. Branch and mesh currents

$$\begin{aligned}
 i1 &= I1 \\
 i2 &= I1 - I2 \\
 i3 &= I2
 \end{aligned}
 \tag{4.40}$$

Example 4.3 Mesh analysis with current sources

Consider the circuit on Figure 14 which contains a current source. The application of the mesh analysis for this circuit does not present any difficulty once we realize that the mesh current of the mesh containing the current source is equal to the current of the current source: i.e.  $I2 = Is$ .

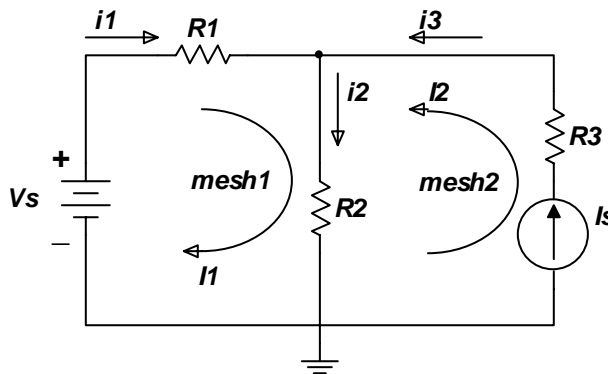


Figure 14. Mesh analysis with a current source.

In defining the direction of the mesh current we have used the direction of the current  $Is$ . We also note that the branch current  $i3 = Is$ .

Applying KVL around mesh1 we obtain

$$I_1 R_1 + (I_1 + I_s) R_2 = V_s \quad (4.41)$$

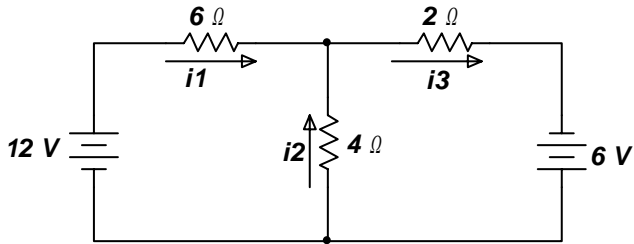
The above equation simply indicates that the presence of the current source in one of the meshes reduces the number of equations in the problem.

The unknown mesh current is

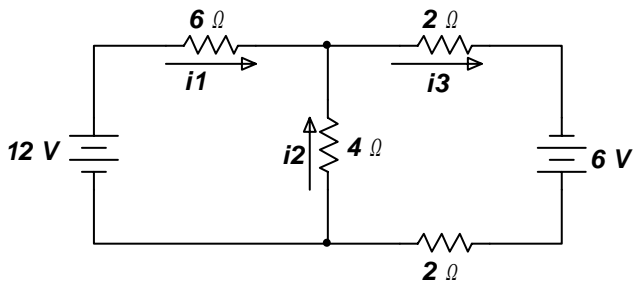
$$I_1 = \frac{V_s - I_s R_2}{R_1 + R_2} \quad (4.42)$$

Practice problems with answers.

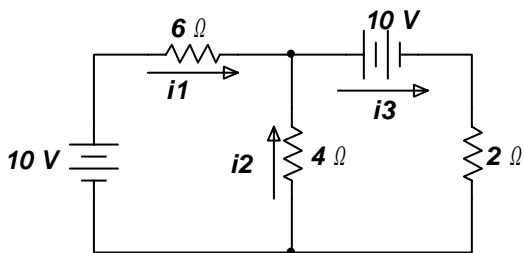
Determine the currents in the following circuits with reference to the indicated direction.



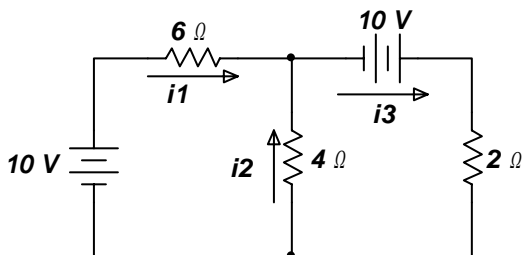
Answer:  $i_1 = 2.180\text{ A}$ ,  $i_2 = 0.270\text{ A}$ ,  $i_3 = 2.450\text{ A}$



Answer:  $i_1 = 1.877\text{ A}$ ,  $i_2 = -0.187\text{ A}$ ,  $i_3 = 1.690\text{ A}$

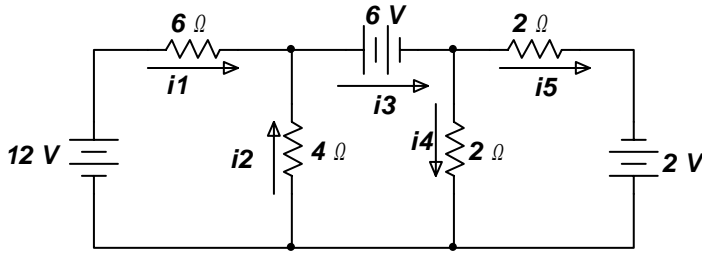


Answer:  $i_1 = 0.455\text{ A}$ ,  $i_2 = -1.820\text{ A}$ ,  $i_3 = -1.36\text{ A}$

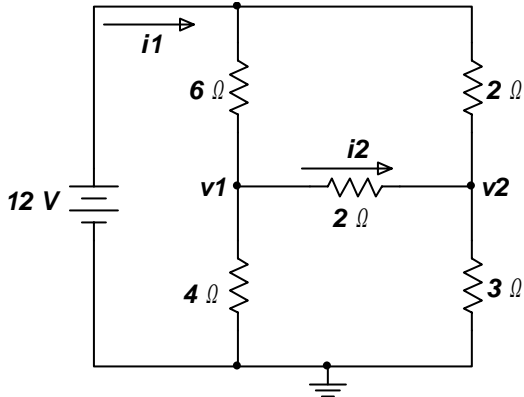


Answer:  $i_1 = 2.270\text{ A}$ ,  $i_2 = 0.909\text{ A}$ ,  $i_3 = 3.180\text{ A}$

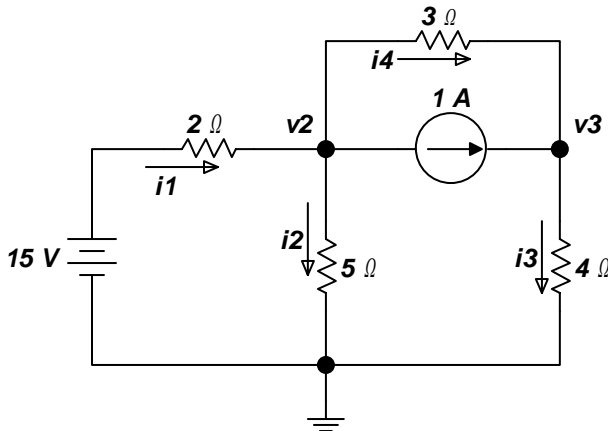




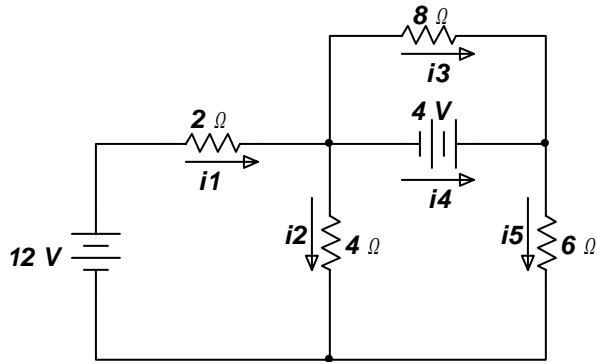
Answer:  $i_1 = 1.180\text{A}$ ,  $i_2 = -1.240\text{A}$ ,  $i_3 = -0.058\text{A}$ ,  $i_4 = -0.529\text{A}$ ,  $i_5 = 0.471\text{A}$



Answer:  $i_1 = 3.690\text{A}$ ,  $i_2 = -0.429\text{A}$ ,  $v_1 = 5.83\text{V}$ ,  $v_2 = 6.69\text{V}$



Answer:  $i_1 = 3.31\text{A}$ ,  $i_2 = 1.68\text{A}$ ,  $i_3 = 1.63\text{A}$ ,  $i_4 = 0.627\text{A}$ ,  $v_2 = 8.39\text{V}$ ,  $v_3 = 6.51\text{V}$



Answer:  $i_1 = 3.09\text{ A}$ ,  $i_2 = 1.45\text{ A}$ ,  $i_3 = -0.50\text{ A}$ ,  $i_4 = 2.14\text{ A}$ ,  $i_5 = 1.64\text{ A}$

Problems:

4.1. For the circuit on Figure P1:

1. Set up the problem for analysis using the nodal method. Indicate the reference node that will make the problem easy to solve.
2. Derive an expression of the voltage  $v_1$  and the current  $i_5$

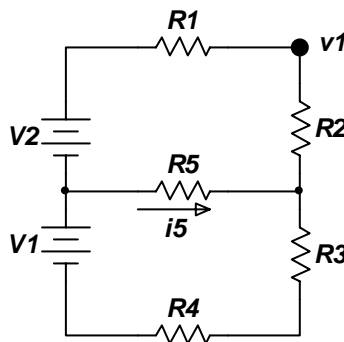


Figure P1

4.2 For the circuit shown on Figure P2 derive the equations for the voltages  $v_1$ ,  $v_2$ ,  $v_3$  using nodal analysis.

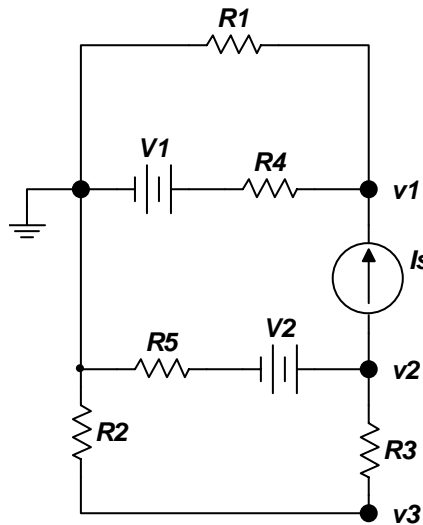


Figure P2

4.3 Repeat problem 4.2 using mesh analysis

4.4 Determine the currents  $i_1, i_2, i_3$  for the circuit on Figure P4 with the reference node as indicated.

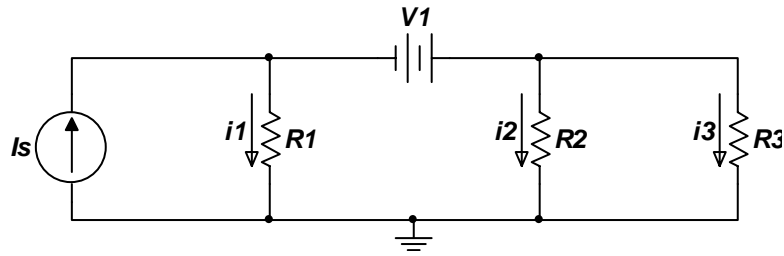


Figure P4

4.5 Determine the currents  $i_1, i_2, i_3$  for the circuit on Figure P5 with the reference node as indicated.

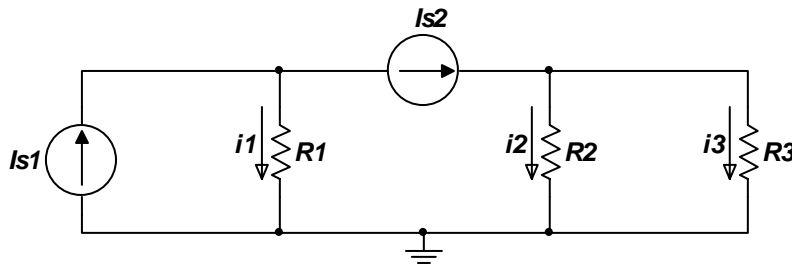


Figure P5

4.6 For the circuit shown on Figure P6:

1. Determine the current  $i_5$  through resistor  $R_5$
2. Derive the condition for which  $\frac{i_5}{I_s} = 0.01$
3. Assume that  $I_s$  is known with an error of  $\delta_s$  and the circuit resistors have a tolerance of  $\delta_R$ . Calculate the uncertainty in  $i_5$
4. If resistor  $R_5$  represents the internal resistance of a measuring device, estimate the relative value of  $R_5$  so that the measurement deviates a maximum of 1% from the ideal.

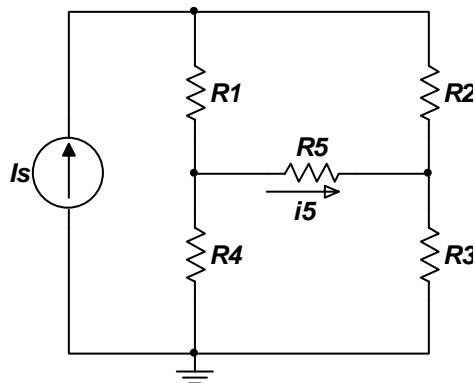


Figure P6