Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 2, April 21, 2010

QUESTION BOOKLET

Your Full Name:	
Recitation Time :	o'clock

- Check that this **QUESTION BOOKLET** has pages numbered up to 7.

- There are **4 problems, weighted as shown**. (The points indicated on the following pages for the various subparts of the problems are our best guesses for now, but may be modified slightly when we get to grading.)

Problem
1 (6 points)
2 (15 points)
3 (15 points)
4 (14 points)
Total (50 points)

Problem 1 (6 points)

For each of the following statements, specify whether the statement is **true** or **false** and give a brief (at most a few lines) justification or counterexample.

(1a) (1.5 points) If X and Y are independent random variables, the unconstrained MMSE estimator $\widehat{Y}$ of Y given $X = x$ is $\widehat{Y} = \mu_Y$.

(1b) (1.5 points) The following random process is strict sense stationary: $x(t) = A$ where A is a continuous random variable with a pdf uniform between ± 1.

(1c) (1.5 points) The following random process is ergodic in the mean: $x(t) = A$ where A is a continuous random variable with a pdf uniform between ± 1.

(1d) (1.5 points) If the input to a stable LTI system is WSS then the output is guaranteed to be WSS.

Problem 2 (15 points)

Suppose $x(t)$ and $v(t)$ are two independent WSS random processes with autocorrelation functions respectively $R_{xx}(\tau)$ and $R_{vv}(\tau)$.

(2a) (3 points) Using $x(t)$ and $v(t)$, show how you would construct a random process $g(t)$ whose autocorrelation function $R_{gg}(\tau)$ is that shown in Eq. (2.1). **Be sure to demonstrate that the resulting process $g(t)$ has the desired autocorrelation function**.

$$R_{gg}(\tau) = R_{xx}(\tau)R_{vv}(\tau) \tag{2.1}$$

For the remainder of this problem assume that $R_{xx}(\tau)$ and $R_{vv}(\tau)$ are known to be

$$R_{xx}(\tau) = 2e^{-|\tau|} \qquad R_{vv}(\tau) = e^{-3|\tau|} \tag{2.2}$$

You can also invoke the Fourier transform identity

$$e^{-\beta|\tau|} \Longleftrightarrow \frac{2\beta}{\beta^2 + \omega^2} \tag{2.3}$$

(2b) (6 points) Let $w(t)$ denote a third WSS random process with autocorrelation function $R_{ww}(\tau) = \delta(\tau)$. Suppose $w(t)$ is the input to a first-order, stable, causal LTI system for which the linear constant coefficient differential equation is

$$\frac{dx(t)}{dt} + ax(t) = bw(t) \tag{2.4}$$

Determine the values of a and b so that the autocorrelation of the output $x(t)$ is $R_{xx}(\tau)$ as given in Eq. (2.2).

(2c) (6 points) Is it possible to have a WSS process $z(t)$ whose autocorrelation function $R_{zz}(\tau) = R_{xx}(\tau) * R_{vv}(\tau)$, i.e., $R_{zz}(\tau)$ is the result of convolving the autocorrelation functions $R_{xx}(\tau)$ and $R_{vv}(\tau)$ given in Eq. (2.2)? If no, why not? If yes, show how to generate $z(t)$ starting from a WSS process $w(t)$ with autocorrelation function $R_{ww}(\tau) = \delta(\tau)$ (you can perform any necessary *causal* LTI filtering operations).

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Problem 3 (15 points)

A DT wide-sense stationary random process $s[n]$ has zero mean and the autocovariance function $C_{ss}[m]$ shown in Figure 3.1. $C_{ss}[m] = 0$ for $|m| \geq 5$.
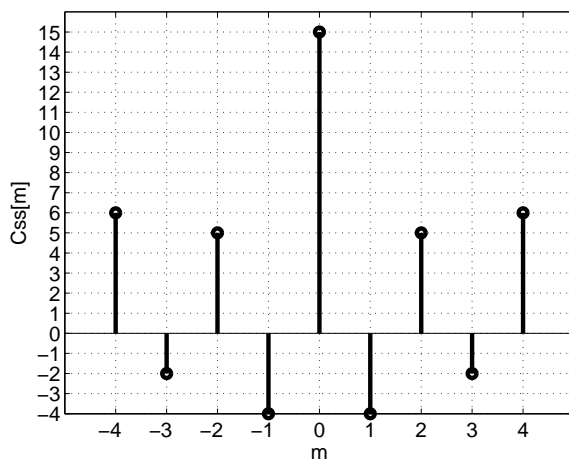

Figure 3.1

The process $s[n]$ is transmitted through a communication channel that introduces noise and a one-step delay. The received process $r[n]$ is related to $s[n]$ as indicated by Eq. (3.1). The noise process $w[n]$ has zero mean, autocovariance function $C_{ww}[m] = 3\delta[m]$. Furthermore, the noise process $w[\cdot]$ is uncorrelated with the process $s[\cdot]$.

$$r[n] = s[n-1] + w[n] \tag{3.1}$$

We wish to design an estimator $\widehat{s}[n]$ for $s[n]$. Throughout this problem the objective is to choose the parameters in the estimator to minimize the mean squared error $\mathcal{E}$ defined as

$$\mathcal{E} \triangleq E\{(s[n] - \widehat{s}[n])^2\}$$

(3a) (2 points) Express the covariance functions $C_{rs}[m]$ and $C_{rr}[m]$ in terms of the given autocovariance functions $C_{ss}[m]$ and $C_{ww}[m]$?

(3b) (3 points) If the estimator is restricted to be of the form

$$\widehat{s}[n] = a_0 r[n] + a_1 r[n-1]$$

will $\mathcal{E}$ be minimized with the parameter values $a_0 = 0$ and $a_1 = 5$?

(3c) (5 points) If the estimator is restricted to be of the form

$$\widehat{s}[n] = a_2 r[n]$$

determine the value of a_2 that minimizes $\mathcal{E}$.

(3d) (5 points) If the estimator is of the form

$$\widehat{s}[n] = \frac{1}{2} r[n - n_o]$$

with $n_o \geq 0$, determine the value of n_o that minimizes $\mathcal{E}$.

Problem 4 (14 points)

Note: Nothing on this problem assumes or requires that you attended today's lecture.
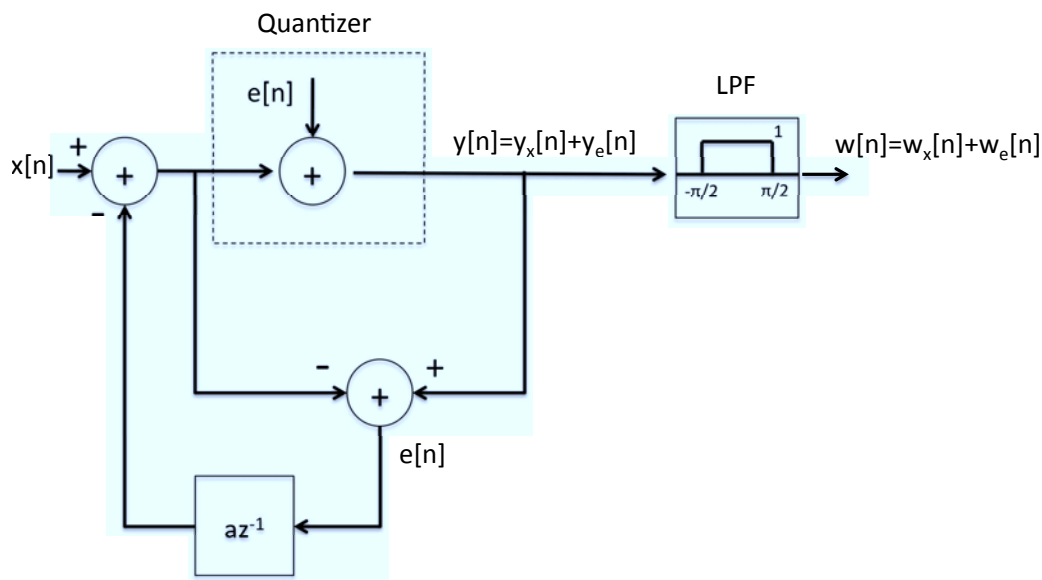

Figure 4.1

D/A converters utilizing oversampled noise shaping often use the structure in Figure 4.1 to quantize the output sequence to a 1-bit bitstream for which D/A conversion is then very simple. Its effectiveness assumes that $x[n]$ is highly oversampled and that most of the error introduced by the quantizer falls outside the signal band.

In Figure 4.1 the quantizer error is modeled as an additive, zero mean i.i.d sequence $e[n]$ that is uniformly distributed in amplitude between $\pm\frac{\triangle}{2}$ at each time instant n. Furthermore, $e[n]$ and $x[n]$ are assumed to be independent. The input $x[n]$ has mean $\mu_x = 0$ and variance $\sigma_x^2 = 1$. Moreover, the input $x[n]$ is bandlimited, i.e., the power spectral density $S_{xx}(e^{j\Omega})$ is zero for $\frac{\pi}{2} < |\Omega| < \pi$.

The output $y[n]$ is composed of two components: $y_x[n]$ due solely to $x[n]$, and $y_e[n]$ due solely to $e[n]$. Similarly, $w[n] = w_x[n] + w_e[n]$. Note that the transfer function from $x[n]$ to

$y_x[n]$ is unity, and the transfer function from $e[n]$ to $y_e[n]$ is $(1 - az^{-1})$.

(4a) (7 points) Determine the power spectral density of $y_e[n]$ as a function of Ω for $|\Omega| < \pi$.

(4b) (7 points) Determine $E\{w_x^2[n]\}$, $E\{w_e^2[n]\}$, and the value of the gain constant "a" that maximizes the signal-to-noise ratio (SNR) of $w[n]$ defined as

$$SNR = \frac{E\{w_x^2[n]\}}{E\{w_e^2[n]\}}$$

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