6.254 : Game Theory with Engineering Applications
Lecture 3: Strategic Form Games - Solution Concepts

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Outline

- Review
- Examples of Pure Strategy Nash Equilibria
- Mixed Strategies and Mixed Strategy Nash Equilibria
- Characterizing Mixed Strategy Nash Equilibria
- Rationalizability

- Reading:
  - Fudenberg and Tirole, Chapters 1 and 2.

Pure Strategy Nash Equilibrium

Definition

(Nash equilibrium) *A (pure strategy) Nash Equilibrium of a strategic game* $\langle \mathcal{I}, (S_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}} \rangle$ *is a strategy profile* $s^* \in S$ *such that for all* $i \in \mathcal{I}$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

- Why is this a "reasonable" notion?
- No player can profitably deviate given the strategies of the other players. Thus in Nash equilibrium, "best response correspondences intersect".
- Put differently, the conjectures of the players are *consistent*: each player i chooses s_i^* expecting all other players to choose s_{-i}^*, and each player's conjecture is verified in a Nash equilibrium.

Example: Second Price Auction

- Second Price Auction (with Complete Information) The second price auction game is specified as follows:
- An object to be assigned to a player in $\{1, .., n\}$.
- Each player has her own valuation of the object. Player i's valuation of the object is denoted v_i. We further assume that $v_1 > v_2 > ... > 0$.
- Note that for now, we assume that everybody knows all the valuations $v_1, \ldots, v_n$, i.e., this is a complete information game. We will analyze the incomplete information version of this game in later lectures.
    - The assignment process is described as follows:
    - The players simultaneously submit bids, $b_1, .., b_n$.
    - The object is given to the player with the highest bid (or to a random player among the ones bidding the highest value).
    - The winner pays the second highest bid.
    - The utility function for each of the players is as follows: the winner receives her valuation of the object minus the price she pays, i.e., $v_i - b_j$; everyone else receives 0.

Second Price Auction (continued)

Proposition

In the second price auction, truthful bidding, i.e., $b_i = v_i$ for all i, is a Nash equilibrium.

Proof: We want to show that the strategy profile $(b_1, .., b_n) = (v_1, .., v_n)$ is a Nash Equilibrium—**a truthful equilibrium**.

- First note that if indeed everyone plays according to that strategy, then player 1 receives the object and pays a price v_2.
- This means that her payoff will be $v_1 - v_2 > 0$, and all other payoffs will be 0. Now, player 1 has no incentive to deviate, since her utility can only decrease.
- Likewise, for all other players $v_i \neq v_1$, it is the case that in order for v_i to change her payoff from 0 she needs to bid more than v_1, in which case her payoff will be $v_i - v_1 < 0$.
- Thus no incentive to deviate from for any player.

Second Price Auction (continued)

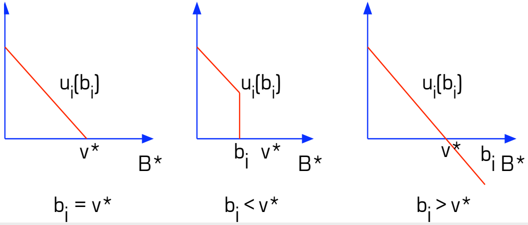
- Are There Other Nash Equilibria? In fact, there are also unreasonable Nash equilibria in second price auctions.
- We show that the strategy $(v_1, 0, 0, ..., 0)$ is also a Nash Equilibrium.
- As before, player 1 will receive the object, and will have a payoff of $v_1 - 0 = v_1$. Using the same argument as before we conclude that none of the players have an incentive to deviate, and the strategy is thus a Nash Equilibrium.
- It can be verified the strategy $(v_2, v_1, 0, 0, ..., 0)$ is also a Nash Equilibrium.
- Why?

Second Price Auction (continued)

- Nevertheless, the truthful equilibrium, where , $b_i = v_i$, is the **Weakly Dominant Nash Equilibrium**
- In particular, truthful bidding, $b_i = v_i$, weakly dominates all other strategies.
- Consider the following picture proof where B^* represents the maximum of all bids excluding player i's bid, i.e.

$$B^* = \max_{j \neq i} b_j,$$

and v^* is player i's valuation and the vertical axis is utility.



Second Price Auction (continued)

- The first graph shows the payoff for bidding one's valuation. In the second graph, which represents the case when a player bids lower than their valuation, notice that whenever $b_i \leq B^* \leq v^*$, player i receives utility 0 because she loses the auction to whoever bid B^*.
- If she would have bid her valuation, she would have positive utility in this region (as depicted in the first graph).
- Similar analysis is made for the case when a player bids more than their valuation.

- An immediate implication of this analysis is that other equilibria involve the play of weakly dominated strategies.

Nonexistence of Pure Strategy Nash Equilibria

- Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- No pure Nash equilibrium.
- How would you play this game?

Nonexistence of Pure Strategy Nash Equilibria

- Example: The Penalty Kick Game.

penalty taker \ goalie	left	right
left	$(-1, 1)$	$(1, -1)$
right	$(1, -1)$	$(-1, 1)$

- No pure Nash equilibrium.
- How would you play this game if you were the penalty taker?
  - Suppose you always show up left.
  - Would this be a "good strategy"?
- Empirical and experimental evidence suggests that most penalty takers "randomize"$\rightarrow$mixed strategies.

Mixed Strategies

- Let Σ_i denote the set of probability measures over the pure strategy (action) set S_i.
  - For example, if there are two actions, S_i can be thought of simply as a number between 0 and 1, designating the probability that the first action will be played.
- We use $\sigma_i \in \Sigma_i$ to denote the mixed strategy of player i, and $\sigma \in \Sigma = \prod_{i \in \mathcal{I}} \Sigma_i$ to denote a mixed strategy profile.
- Note that this implicitly assumes that **players randomize independently**.
- We similarly define $\sigma_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j$.
- Following von Neumann-Morgenstern expected utility theory, we extend the payoff functions u_i from S to Σ by

$$u_i(\sigma) = \int_S u_i(s) d\sigma(s).$$

Mixed Strategy Nash Equilibrium

Definition (Mixed Nash Equilibrium)

A mixed strategy profile σ^* *is a (mixed strategy) Nash Equilibrium if for each player* i*,*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \qquad \text{for all } \sigma_i \in \Sigma_i.$$

- It is sufficient to check only *pure* strategy "deviations" when determining whether a given profile is a (mixed) Nash equilibrium.

Proposition

A mixed strategy profile σ^* *is a (mixed strategy) Nash Equilibrium if and only if for each player* i*,*

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \qquad \text{for all } s_i \in S_i.$$

Mixed Strategy Nash Equilibria (continued)

- We next present a useful result for characterizing mixed Nash equilibrium.

Proposition

Let $G = \langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ be a finite strategic form game. Then, $\sigma^* \in \Sigma$ is a Nash equilibrium if and only if for each player $i \in \mathcal{I}$, every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*.

Proof idea: If a mixed strategy profile is putting positive probability on a strategy that is not a best response, then shifting that probability to other strategies would improve expected utility.

Mixed Strategy Nash Equilibria (continued)

- It follows that every action in the support of any player's equilibrium mixed strategy yields the same payoff.
- Note: this characterization result extends to **infinite games**: $\sigma^* \in \Sigma$ is a Nash equilibrium if and only if for each player $i \in \mathcal{I}$,
  (i) no action in S_i yields, given σ^*_{-i}, a payoff that exceeds his equilibrium payoff,
  (ii) the set of actions that yields, given σ^*_{-i}, a payoff less than his equilibrium payoff has σ^*_i-measure zero.

Examples

Example: Matching Pennies.

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- Unique mixed strategy equilibrium where both players randomize with probability $1/2$ on heads.

Example: Battle of the Sexes Game.

Player 1 \ Player 2	ballet	football
ballet	$(2, 1)$	$(0, 0)$
football	$(0, 0)$	$(1, 2)$

- This game has two pure Nash equilibria and a mixed Nash equilibrium $\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}) \right)$.

Strict Dominance by a Mixed Strategy

Player 1 \ Player 2	*Left*	*Right*
U	$(2,0)$	$(-1,0)$
M	$(0,0)$	$(0,0)$
D	$(-1,0)$	$(2,0)$

- Player 1 has no pure strategies that strictly dominate M.
- However, M is strictly dominated by the mixed strategy $(\frac{1}{2}, 0, \frac{1}{2})$.

Definition (Strict Domination by Mixed Strategies)

An action s_i is **strictly dominated** *if there exists a mixed strategy $\sigma_i' \in \Sigma_i$ such that $u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$, for all $s_{-i} \in S_{-i}$.*

Remarks:

- Strictly dominated strategies are never used with positive probability in a mixed strategy Nash Equilibrium.
- However, as we have seen in the Second Price Auction, weakly dominated strategies can be used in a Nash Equilibrium.

Iterative Elimination of Strictly Dominated Strategies–Revisited

- Let $S_i^0 = S_i$ and $\Sigma_i^0 = \Sigma_i$.
- For each player $i \in \mathcal{I}$ and for each $n \geq 1$, we define S_i^n as

$$S_i^n = \{s_i \in S_i^{n-1} \mid \nexists \quad \sigma_i \in \Sigma_i^{n-1} \text{ such that } u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}^{n-1}\}.$$

- Independently mix over S_i^n to get Σ_i^n.
- Let $D_i^\infty = \cap_{n=1}^\infty S_i^n$.
- We refer to the set D_i^∞ as the set of strategies of player i that survive iterated strict dominance.

Rationalizability

- In the Nash equilibrium concept, each player's action is optimal conditional on the **belief** that the other players also play their Nash equilibrium strategies.
  - The Nash Equilibrium strategy is optimal for a player given his belief about the other players strategies, and this belief is correct.
- We next consider a different solution concept in which a player's belief about the other players' actions is not assumed to be correct, but rather, simply constrained by rationality.

Definition

A belief *of player i about the other players' actions is a probability measure $\sigma_{-i} \in \prod_{j \neq i} \Sigma_j$ (recall that Σ_j denotes the set of probability measures over S_j, the set of actions of player j).*

Rationality

- Rationality imposes two requirements on strategic behavior:
  (1) Players maximize with respect to some beliefs about opponent's behavior (i.e., they are rational).
  (2) Beliefs have to be consistent with other players being rational, and being aware of each other's rationality, and so on (but they need not be correct).
- Rational player i plays a best response to some belief σ_{-i}.
- Since i thinks j is rational, he must be able to rationalize σ_{-i} by thinking every action of j with $\sigma_{-i}(s_j) > 0$ must be a best response to some belief j has.

  $\vdots$

- Leads to an infinite regress: "I am playing strategy σ_1 because I think player 2 is using σ_2, which is a reasonable belief because I would play it if I were player 2 and I thought player 1 was using σ_1', which is a reasonable thing to expect for player 2 because σ_1' is a best response to σ_2', ...

Example

Consider the game (from [Bernheim 84]),

	b_1	b_2	b_3	b_4
a_1	$0, 7$	$2, 5$	$7, 0$	$0, 1$
a_2	$5, 2$	$3, 3$	$5, 2$	$0, 1$
a_3	$7, 0$	$2, 5$	$0, 7$	$0, 1$
a_4	$0, 0$	$0, -2$	$0, 0$	$10, -1$

There is a unique Nash equilibrium (a_2, b_2) in this game, i.e., the strategies a_2 and b_2 rationalize each other. Moreover, the strategies a_1, a_3, b_1, b_3 can also be rationalized:

- Row will play a_1 if Column plays b_3.
- Column will play b_3 if Row plays a_3.
- Row will play a_3 if Column plays b_1.
- Column will play b_1 if Row plays a_1.

However b_4 cannot be rationalized, and since no rational player will play b_4, a_4 can not be rationalized.

Never-Best Response Strategies

Example

Consider the following game:

	Q	F
Q	4, 2	0, 3
X	1, 1	1, 0
F	3, 0	2, 2

- *It can be seen that* F *can be rationalized.*
  - *If player* 1 *believes that player 2 will play* F*, then playing* F *is rational for player 1, etc.*
- *However, playing* X *is* never a best response, *regardless of what strategy is chosen by the other player, since playing* F *always results in better payoffs.*
- *A strictly dominated strategy will never be a best response, regardless of a player's beliefs about the other players' actions.*

Never-Best Response and Strictly Dominated Strategies

Definition

A pure strategy s_i *is a* never-best response *if for all beliefs* σ_{-i} *there exists* $\sigma_i \in \Sigma_i$ *such that*

$$u_i(\sigma_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i}).$$

- As shown in the preceding example, a strictly dominated strategy is a never-best response.
- Does the converse hold? Is a never-best response strategy strictly dominated?
- The following example illustrates a never-best response strategy which is not strictly dominated.

Example

Consider the following three-player game in which all of the player's payoffs are the same. Player 1 chooses A or B, player 2 chooses C or D and player 3 chooses M_i for $i = 1, 2, 3, 4$.

M_1:

	C	D
A	8	0
B	0	0

M_2:

	C	D
A	4	0
B	0	4

M_3:

	C	D
A	0	0
B	0	8

M_4:

	C	D
A	3	3
B	3	3

- We first show that playing M_2 is never a best response to any mixed strategy of players 1 and 2.
- Let p represent the probability with which player 1 chooses A and let q represent the probability that player 2 chooses C.
- The payoff for player 3 when she plays M_2 is

$$u_3(M_2, p, q) = 4pq + 4(1-p)(1-q) = 8pq + 4 - 4p - 4q$$

Example

- Suppose, by contradiction, that this is a best response for some choice of p, q. This implies the following inequalities:

$$
\begin{aligned}
8pq + 4 - 4p - 4q &\geq u_3(M_1, p, q) = 8pq \\
&\geq u_3(M_3, p, q) = 8(1-p)(1-q) = 8 + 8pq - 8(p + q \\
&\geq u_3(M_4, p, q) = 3
\end{aligned}
$$

- By simplifying the top two relations, we have the following inequalities:

$$
\begin{aligned}
p + q &\leq 1 \\
p + q &\geq 1
\end{aligned}
$$

Thus $p + q = 1$, and substituting into the third inequality, we have $pq \geq 3/8$. Substituting again, we have $p^2 - p + \frac{3}{8} \leq 0$ which has no positive roots since the left side factors into $(p - \frac{1}{2})^2 + (\frac{3}{8} - \frac{1}{4})$.

- On the other hand, by inspection, we can see that M_2 is not strictly dominated.

Rationalizable Strategies

Iteratively eliminating never-best response strategies yields rationalizable strategies.

- Start with $\tilde{S}_i^0 = S_i$.
- For each player $i \in \mathcal{I}$ and for each $n \geq 1$,

$$\tilde{S}_i^n = \{s_i \in \tilde{S}_i^{n-1} \mid \exists \quad \sigma_{-i} \in \prod_{j \neq i} \tilde{\Sigma}_j^{n-1} \text{ such that}$$
$$u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \quad \text{for all } s_i' \in \tilde{S}_i^{n-1}\}.$$

- Independently mix over $\tilde{S}_i^n$ to get $\tilde{\Sigma}_i^n$.
- Let $R_i^\infty = \cap_{n=1}^\infty \tilde{S}_i^n$. We refer to the set R_i^∞ as the set of rationalizable strategies of player i.

Rationalizable Strategies

- Since the set of strictly dominated strategies is a strict subset of the set of never-best response strategies, set of rationalizable strategies represents a further refinement of the set of strategies that survive iterated strict dominance.
- Let NE_i denote the set of pure strategies of player i used with positive probability in any mixed Nash equilibrium.
- Then, we have

$$NE_i \subseteq R_i^\infty \subseteq D_i^\infty,$$

  where R_i^∞ is the set of rationalizable strategies of player i, and D_i^∞ is the set of strategies of player i that survive iterated strict dominance.

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