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Poisson Processes

1 Counting processes

A stochastic process $N(t),\ t \geq 0$ is said to be a counting process if $N(t)$ satisfies the following properties:

1. $N(t) \geq 0$.
2. $N(t)$ is integer valued.
3. If $s < t$, then $N(s) \leq N(t)$.

Intuitively, $N(t)$ represents the number of events that have occurred up to time t.

A counting process is said to possess independent increments if $a_1 \leq a_2 \leq \cdots \leq a_k$ implies that the random variables $N(a_2)-N(a_1), N(a_3)-N(a_2), \ldots, N(a_k)-N(a_{k-1})$. Intuitively, the number of events occurring in one interval should be independent of the number of events occurring in another interval, provided the intervals are disjoint.

A counting process is said to possess stationary increments if $N(s+t) - N(s)$ depends only on t. Intuitively, the number of events that occur in an interval depends only on its length.

2 Poisson processes

A counting process is said to be Poisson with rate $\lambda > 0$ if it has the following properties:

1. $N(0) = 0$.
2. The process has stationary and independent intervals.
3. $P(N(h) = 1) = \lambda h + o(h)$.

4. $P(N(h) \geq 2) = o(h)$.

Poisson processes may be defined in a different way. A process is said to be Poisson with rate $\lambda > 0$, if

1. $N(0) = 0$.

2. The process has independent increments.

3. For all $s, t \geq 0$,

$$P(N(s+t) - N(s) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Claim: The two definitions of a Poisson process are equivalent.

Proof: That the second definition implies the first follows immediately from the taylor series of $e^{-\lambda t}$. To show that the first definition implies the second, we argue as follows.

Lets use the shorthand $P_n(t) = P(N(t) = n)$. First, let's derive the expression for $P_0(t)$. The assumptions of independence and stationary increments imply

$$P_0(t+h) = P_0(t)P_0(h) = P_0(t)(1 - \lambda h + o(h)),$$

so

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0 + \frac{o(h)}{h},$$

and taking the limit as $h \to 0$, we get[1]

$$P_0' = -\lambda P_0.$$

The solution of this ode is $P_0(t) = Ce^{-\lambda t}$, and since $P_0(0) = 1$, we get $P_0(t) = Ce^{-\lambda t}$.

Now for $n \geq 1$, we have

$$P_n(t+h) = P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h),$$

which gives

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t),$$

[1] It could be argued that we have only shown that the derivative from the right satisfies the ode above. However, one can repeat the same argument beginning with $P_0(t) = P_0(t-h)P_0(h)$, to get the same fact for the left derivative. We omit the details.

so

$$P_n' = -\lambda P_n + \lambda P_{n-1}.$$

The trick is to write this as

$$\frac{d}{dt}(e^{\lambda t} P_n) = e^{\lambda t} \lambda P_{n-1}.$$

With this formula in place, lets prove that $P_n(t) = e^{-\lambda t}(\lambda t)^n/n!$ by induction. We know this is true for $n = 0$. Assuming its true for n, we have

$$\frac{d}{dt}(e^{\lambda t} P_{n+1}(t)) = e^{\lambda t} \lambda e^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

or

$$e^{\lambda t} P_{n+1}(t) = \frac{\lambda^{n+1} t^{n+1}}{(n+1)!} + C,$$

or

$$P_{n+1}(t) = e^{-\lambda t} \frac{\lambda^{n+1} t^{n+1}}{(n+1)!} + C e^{-\lambda t},$$

and since $P_n(0) = 0$, we get $C = 0$. This completes the proof. □

Poisson processes can also be characterized by their interarrival times. Let T_k be the time between the $k - 1$st and kth arrival. What is the distribution of the T_ks?

Clearly,

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t},$$

so T_1 is exponentially distributed with parameter λ. Moving on,

$$P(T_2 > t | T_1 = s) = P(N(s + t) - N(s) = 0 | T_1 = s),$$

and now by independent increments, $N(s + t) - N(s)$ is independent of the event $T_1 = s$, so

$$P(T_2 > t | T_1 = s) = e^{-\lambda t},$$

so T_2 is also exponentially distributed with parameter λ and independent of T_1. Proceeding this way, we get that all of the T_i are iid exponentials with parameter λ.

3 Another definition of the Poisson process

Let S_i be the arrival times of a Poisson process, i.e.

$$\begin{aligned} S_1 &= T_1 \\ S_2 &= T_1 + T_2 \\ S_3 &= T_1 + T_2 + T_3 \\ \vdots & \quad \vdots \quad \vdots \end{aligned}$$

Claim: Conditioned on $N(t) = n$, the distribution of $S_1, \ldots, S_n$ is the same as the distribution of order statistics of $U[0, t]$ random variables.

Remark: This gives another view of the poisson process. We can fix time t, draw n from a poisson distribution with parameter λt, and then generate $S_1, \ldots, S_n$ as order statistics of uniform random variables on $[0, t]$.

Proof: Suppose $t_1 < t_2 < \cdots < t_n$ are points in $(0, t)$. Pick h to be small enough so that $t_i + h < t_{i+1}$. Consider the the probability,

$$P(S_1 \in [t_1, t_1 + h], S_2 \in [t_2, t_2 + h], \ldots, S_n \in [t_n, t_n + h], N(t) = n)$$

This is the same as the probability of exactly one arrival in each interval $[t_i, t_i + h]$ and no arrivals elsewhere in $[0, t]$. So,

$$\begin{aligned} P(S_1 \in [t_1, t_1 + h], S_2 \in [t_2, t_2 + h], \ldots, S_n \in [t_n, t_n + h] \mid N(t) = n) &= \frac{(\lambda h e^{-\lambda h})^n e^{-\lambda(t - nh)}}{e^{-\lambda t} \lambda^n t^n / n!} \\ &= \frac{n!}{t^n} h^n \end{aligned}$$

and this implies that the density of $S_1, \ldots, S_n$ conditioned on $N(t) = n$ is $n!/t^n$[2] $\square$

[2]If you would like to make the last step more precise, one can argue as foolows. Observe that we have two probability measures on R^n: $P_1(A) = P((S_1, \ldots, S_n) \in A)$, and $P_2(A) = \int_A f$, where $f(x_1, \ldots, x_n) = n!/t^n$ whenever $0 < x_1 < x_2 \ldots < x_n < t$, and 0 elsewhere. We want to show that these two measures are the same everywhere. For simplicity, consider the case when $n = 2$. Then, we have shown that these two measures are the same on rectangles of the form $[a, b] \times [c, d]$ where $b \leq c$. They must also be the same on rectangles of the form $[a, b] \times [c, d]$ when $a \geq d$ - the probability is 0 in both cases. Its not hard to see these two facts imply the two measures must be the same on all rectangles, and consequently on all Borel sets.

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