MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Recitation 11

1 On taking limits of random variables

Consider the following problem. Suppose

$$P(B|A) \geq \frac{1}{2} \text{ and } P(C|B) \geq \frac{1}{2}.$$

Is it necessarily true that

$$P(C|A) \geq \frac{1}{4}?$$

Solution: No; let X be a uniform random variable in the interval $[0, 4]$. Let $A = \{X \in [0, 2]\}, B = \{X \in [1, 3]\}, C - \{X \in [2, 4]\}$. Then, $P(B|A) = P(C|B) = 1/2$, but $P(C|A) = 0$. □

In fact, we can strengthen this example. Indeed, suppose X is uniform over $[0, K]$, and $A = \{X \in [0, 1]\}, B = \{X \in [0, K]\}, C = \{X \in [1, K]\}$. Then $P(B|A) = 1, P(C|A) = 0$, and $P(C|B) = (K - 1)/K$. If we pick K large, $P(C|B)$ approaches 1. So it is quite possible that

$$P(B|A) = 1,\ P(C|B) = 1 - \epsilon, P(C|A) = 0,$$

regardless of how small $\epsilon > 0$ is.

It is somewhat surprising therefore that it is not possible to have

$$P(B|A) = 1, P(C|B) = 1, P(C|A) = 0.$$

Indeed, let us write $A \sqsubset B$ if $A \cap B^c$ has measure 0. Observe that $P(B|A) = 1$ is equivalent to[1]

$$A \sqsubset B.$$

So the conditions $P(B|A) = 1$ and $P(C|B) = 1$ can be rewritten as

$$A \sqsubset B, B \sqsubset C,$$

[1]Provided $P(A) > 0$, which we implicitly assume.

which necessarily implies $A \sqsubset C$. Indeed,

$$P(A \cap C^c) = P(A \cap B \cap C^c) + P(A \cap B^c \cap C^c) = 0 + 0,$$

the first term being a subset of the measure-0 set $B \cap C^c$, and the second set being a subset of the measure-0 set $A \cap B^c$. So $A \sqsubset C$ and thus $P(C|A) = 1$.

What is the source of this discontinuity? What happens if we simply let $K \to \infty$ in our counterexample? Unfortunately, X was defined to be uniform over $[1, K]$, and when we let $K \to \infty$, we do not get a random variable.

2 Convergence of densities vs convergence of distributions

Suppose f_n is the density of the random variable X_n, and

$$f_n \to f,$$

pointwise. It does not follow that f is the density of a random variable. As we had just argued in the previous section,

$$\frac{1}{n} 1_{[0,n]} \to 0,$$

and of course the zero function is not a valid density. Another example is $1_{[n,n+1]}$ which also converges to the zero function.

But suppose $f_n \to f$ everywhere, and moreover f_n, f are all valid densities. What is the relationship of this convergence to the convergence of the distributions F_n and F?

We will prove the following two statements.

Claim 1: It is possible that $F_n \to F$ everywhere, that F_n has density f_n, F has density f, and yet nowhere does f_n converge to f.

Claim 2: If $f_n \to f$, and f, f_n are valid distributions, then $F_n \to F$ everywhere.

Proof of Claim 1: Break up the the interval $[0, 1]$ into 2^i intervals

$$[0, 1] = [0, \frac{1}{2^i}] \cup [\frac{1}{2^i}, \frac{2}{2^i}] \cup \cdots \cup [\frac{2^i - 1}{2^i}, 1],$$

and let f_i be 2 on the first, third, ... of these intervals and 0 on the second, fourth, ... of them.

If F is the cdf of the $U[0,1]$ distributions, then its not hard to see that $F_n \to F$. Indeed,

$$\max_{x\in R} |F_n(x) - F(x)| = \frac{1}{2^i}.$$

On the other hand, $f = 1_{[0,1]}$, so f_n does not approach f anywhere. □

Proof of Claim 2: Define $g_i = f_i - f$, and let $g_i = g_i^+ - g_i^-$ be the standard decomposition of g_i into positive and negative parts. Since f_i converges to f almost everywhere, we have that g_i^+ and g_i^- both converge to 0 almost everywhere.

Since f_i is a PDF, it is nonnegative almost everywhere; and therefore, $g_i \geq -f$ almost everywhere, which in turn implies $g_i^- \leq f$ almost everywhere. So g_i^- is upper bounded by an integrable function. Thus we can interchange limit and integration when it comes to g_i^-, and in particular

$$\lim_i \int_{-\infty}^{+\infty} g_i^- = \int_{-\infty}^{+\infty} \lim_i g_i^- = 0.$$

But since $\int_{-\infty}^{+\infty} f_i = 1 = \int_{-\infty}^{+\infty} f$, it follows that

$$\int_{-\infty}^{+\infty} g_i^+ = \int_{-\infty}^{+\infty} g_i^-,$$

and therefore

$$\lim_i \int_{-\infty}^{+\infty} g_i^+ = \lim_i \int_{-\infty}^{+\infty} g_i^- = 0.$$

Now passing from integrals involving g_i^+ and g_i^- to integrals involving g:

$$\lim_i \int_{-\infty}^{+\infty} |g_i| = \lim_i [\int_{-\infty}^{+\infty} g_i^+ + \int_{-\infty}^{+\infty} g_i^-] = 0.$$

Now we use the last equation to prove convergence in distribution:

$$\begin{aligned} |F(x) - F_n(x)| &\leq \int_{-\infty}^{x} |f(u) - f_n(u)| du \\ &\leq \int_{-\infty}^{+\infty} |f(u) - f_n(u)| du \\ &= \int_{-\infty}^{+\infty} |g_n(u)| du \end{aligned}$$

and we have just shown that the last expression approaches 0.

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