1 Compositions of n

1. What is the number of ways to write n as an ordered sum of k *positive* integers $n_1, \ldots, n_k$?

Solution: Imagine lining up m points. A partitioning of n points in k parts containing at *least one point* requires $k-1$ separations, and there are $n-1$ choices for the placement of the partitions. Therefore, the answer is $\binom{n-1}{k-1}$.

2.What is the number of ways to write n as an ordered sum of k *nonnegative* integers $n_1, \ldots, n_k$?

Solution: Note that the number of ways to write

$$n_1 + n_2 + \cdots + n_k = n, \qquad n_i \geq 0$$

is the same as the number of ways to write

$$(n_1 + 1) + (n_2 + 1) + \cdots + (n_k + 1) = n + k, \qquad n_i \geq 0,$$

which is the same as the number of ways to write

$$\hat{n}_1 + \hat{n}_2 + \cdots + \hat{n}_k = n + k, \qquad \hat{n}_i \geq 1.$$

We now apply part 1 to get that the answer is $\binom{n+k-1}{k-1}$.

3. What is the number of ways to express a nonnegative integer as an ordered sum of *positive* integers. For example,

$$4 = \left\{ \begin{array}{ll} 1+1+1+1 & 3+1 \\ 2+1+1 & 1+3 \\ 1+2+1 & 2+2 \\ 1+1+2 & 4 \end{array} \right\}.$$

Solution: Consider all ways to write n as a sum of positive integers

$$n = a_1 + a_2 \cdots,$$

and map each of them into **two** ways to write $n+1$ as a sum of positive integers:

$$\begin{aligned} n+1 &= 1 + a_1 + a_2 + \cdots, \\ n+1 &= (a_1 + 1) + a_2 + \cdots \end{aligned}$$

It is easy to check that every composition of $n+1$ can be obtained in this way from a composition of n; and that images of two different compositions of n are always differen. Letting $f(n)$ be the number of compositions of n, we have just showed that $f(n+1) = 2f(n)$. Since $f(1) = 1$, we have that $f(n) = 2^{n-1}$.

Here is a different proof. Imagine lining up n points. There are $n-1$ possible points where we could decide to cut this line between points. There are two choices (to cut or not to cut) for every place, and every selection of these choices yields a unique composition of n. Moreover, every composition corresponds to exactly one selection of choices. Thus the answer is 2^{n-1}.

2 Measurability of random variables

2.1 Definition

Remember that a function h between two measurable space $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ is measurable if and only if that $\forall A' \in \mathcal{F}'$, $h^{-1}(A') \in \mathcal{F}$.

The following statement provides a method to show that a function is measurable.

Let $\mathcal{S}'$ a collection of sets of Ω' such that $\sigma(\mathcal{S}') = \mathcal{F}'$. Then, h is measurable if and only if $h^{-1}(S')$ is $\mathcal{F}$-measurable for all $S' \in \mathcal{S}'$.

Since the Borel σ-algebra is generated $\{(-\infty, c],\ c \in R\}$, it is enough to check that $X^{-1}\left((-\infty, c]\right)$ is $\mathcal{F}$-measurable for all $c \in R$.

2.2 $\min(X, Y)$, $\inf X_n$ are random variables

Let $(X_n), Y$ be random variables on $(\Omega, \mathcal{F})$

- $\min(X, Y)$ is a random variable. Indeed, fix $c \in R$. $M_c = \{\omega \in \Omega | \min(X(\omega), Y(\omega))) \leq c\} = \{\omega \in \Omega | X(\omega) \leq c\} \bigcup \{\omega \in \Omega | Y(\omega) \leq c\}$.
- $\inf_n X_n$ is a random variable since $\{\inf_n X_n \geq c\} = \bigcap_n \{X_n \geq c\}$. Observe that we could not have used the corresponding strict inequality here.

2.3 Continuity implies measurability

Let $f : R \to R$ is a continuous function. We will show that f is $\mathcal{B}(\Omega)$-measurable, where $\mathcal{B}(\Omega)$ is the Borel σ-field of Ω.

Proof: We need to show that $f^{-1}((a, b))$ is measurable for every $a < b$. In fact, we will prove that $f^{-1}((a, b))$ is open. Then, $f^{-1}((a, b))$ must also be measurable.

Let $S = f^{-1}((a, b))$, and take $x \in S$, and let $y = f(x)$. By definition, $y \in (a, b)$. Since (a, b) is an open interval, there must exist some $\varepsilon > 0$ such that $(y - \varepsilon, y + \varepsilon) \subseteq (a, b)$. Now by the definition of continuity at a point x, we know that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $\|\hat{x} - x\| < \delta$, we have $|f(x) - f(\hat{x})| < \varepsilon$. In particular, this means that $f(\hat{x}) \in (y - \varepsilon, y + \varepsilon)$, and therefore $\{x \in R^n | \|\hat{x} - x\| < \delta\}(x - \delta, x + \delta) \subseteq S$. Thus $f^{-1}((a, b))$ is open, and in particular it is measurable.

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