MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2007 — 6.436J/15.085J

Midterm exam, 7-9pm, (120 mins/70 pts) — 10/23/07

Possibly useful facts:
(a) If X is uniform on $[a, b]$, then the variance of X is $(b-a)^2/12$.
(b) $\sum_{n=1}^{\infty} 1/n^{\alpha}$ is infinite when $\alpha \leq 1$, and finite when $\alpha > 1$.

Problem 1: (10 points)
Let $\mathcal{F}$ be a σ-field of subsets of a sample space Ω. Let H be a subset of Ω that does not belong to $\mathcal{F}$. Consider the collection $\mathcal{G}$ of all sets of the form

$$(H \cap A) \cup (H^c \cap B),$$

where $A \in \mathcal{F}$ and $B \in \mathcal{F}$.

(a) Show that if $A \in \mathcal{F}$, then $(H \cap A) \in \mathcal{G}$.

(b) Show that $\mathcal{G}$ is a σ-field.

Solution:

For part (a), we can simply take $B = \emptyset$, which must be a member of $\mathcal{F}$.

For part (b), let us check the σ-field requirements. We can take $A = B = \emptyset$ to get that $\emptyset \in \mathcal{G}$. To show complements are in $\mathcal{G}$, let $K \in \mathcal{G}$. Then, $K = (H \cap A) \cup (H^c \cap B)$ for some $A, B \in \mathcal{F}$. Define $K^* = (H \cap A^c) \cup (H^c \cap B^c)$. Clearly, $K^* \in \mathcal{G}$.

Now observe that $K \cup K^* = \Omega$. Indeed, one is either in H or in H^c; if one is in H, we can further say one is either in A or A^c; if one is in H^c, we can further say one is either in B or B^c. Either way, one belongs to one of the four clauses in $K \cup K^*$.

Next, observe that $K \cap K^* = \emptyset$. Indeed, if one is in H then one cannot be in both K and K^* since this means belong to both A and A^c. Similarly, if one is in H^c, one cannot belong to both K and K^* since this means belonging to both B and B^c.

We conclude that K^* is the complement of K, and therefore $\mathcal{G}$ is closed under complementation.

Now it remains to show that countable unions are in $\mathcal{G}$. Let $D_i \in \mathcal{G}$; then $D_i = (H \cap A_i) \cup (H^c \cap B_i)$ for some $A_i, B_i \in \mathcal{F}$. Then,

$$\begin{aligned} \cup_{i=1}^{\infty} D_i &= \cup_{i=1}^{\infty}(H \cap A_i) \cup \cup_{i=1}^{\infty}(H^c \cap B_i) \\ &= (H \cap \cup_{i=1}^{\infty} A_i) \cup (H^c \cap \cup_{i=1}^{\infty} B_i) \end{aligned}$$

which must belong to $\mathcal{G}$ since $\cup_{i=1}^{\infty} A_i$ and $\cup_{i=1}^{\infty} B_i$ are both in $\mathcal{F}$.

Problem 2: (10 points)
Let X_n be i.i.d. random variables, defined on the same probability space that

are exponentially distributed, with PDF $f(x) = e^{-x}$, $x \geq 0$. Let c be a positive constant, and consider the event A that

$$\text{“}X_n \geq c \log n \text{ for infinitely many values of } n\text{”}$$

Find a necessary and sufficient condition for $\mathbf{P}(A)$ to be equal to 1.

Solution:

First, note that if X_n has PDF e^{-x}, its CDF is $1-e^{-x}$ and $P(X_n \geq x) = e^{-x}$. Defining

$$A_n = X_n \geq c \log n,$$

a necessary and sufficient condition for the statement “A_n occurs infinitely often” is, by independence of A_n and the Borel-Cantelli lemmas,

$$\sum_{n=1}^{\infty} P(A_n) = \infty.$$

However,

$$P(A_n) = e^{-c \log n} = \frac{1}{n^c},$$

so that applying “possibly useful fact (b)” gives us the necessary and sufficient condition $c \leq 1$.

Problem 3: (10 points)
Let X_1, X_2, X_3 be independent random variables, uniformly distributed on [0,1]. Let Y be the median of X_1, X_2, X_3 (that is the middle of the three values). Find the conditional CDF of X_1, given the event $Y = 1/2$. Under this conditional distribution, is X_1 continuous? Discrete?

Solution:

There are three possibilities; X_1 is either the smallest, the median, or the largest. Each of these possibilities occurs with probability $1/3$.

If X_1 is the smallest, then given that the median is $1/2$, its distribution is uniform in $[0, 1/2]$. If X_1 is the median, it is equal to $1/2$ with probability 1. If X_1 is the largest, then conditioned on the median being $1/2$ it is uniform in $[1/2, 1]$.

Therefore, $P(X_1 \leq x) = 2x/3$, if $x < 1/2$; $2/3$ if $x = 1/2$; and $2/3+(1/3)(x-1/2)/(1/2)$ if $x > 1/2$.

For part (b), conditioned on $Y = 1/2$ we have that X_1 is neither continuous nor discrete. It has a jump on $x = 1/2$, so that it cannot have a density, and thus it is not continuous; and there are infinitely many values it can take, so that it is not discrete.

Problem 4: (10 points)
Let X and Y be independent random variables, uniformly distributed on [0,2].

(a) Find the mean and variance of XY.

(b) Calculate the probability $\mathbf{P}(XY \le 1)$.

Solution:

Note that the expectation of X and Y is both 1, and, applying "possibly useful fact (a)", the variance of both X and Y is 1/3, which implies $E[X^2] = E[Y^2] = 4/3$.

By independence,

$$E[XY] = E[X]E[Y] = 1 \cdot 1 = 1.$$

To compute the variance, we argue again by independence

$$E[(XY)^2] = E[X^2]E[Y^2] = \frac{16}{9},$$

and therefore

$$var(XY) = \frac{16}{9} - 1 = \frac{7}{9}.$$

Finally,

$$P(XY \le 1) \quad = \quad P(Y \le \frac{1}{X}) = \int_0^2 F_Y(\frac{1}{x}) f_X(x)dx$$

Now observe that when $X \le 1/2$, then $Y \le 1/X$ with probability 1; and when $X \in (1/2, 2)$, then $Y \le 1/X$ with probability $(1/X)/2$. Thus,

$$\begin{aligned} P(XY \le 1) &= \int_0^{1/2} 1 \cdot \frac{1}{2}dx + \int_{1/2}^2 \frac{1}{2x}\frac{1}{2}dx \\ &= \frac{1}{4} + \frac{1}{4}(\log 2 - \log\frac{1}{2}) \\ &= \frac{1}{4} + \frac{\log 2}{2} \end{aligned}$$

Problem 5: (10 points)
A standard card deck (52 cards) is distributed to two persons: 26 cards to each person. All partitions are equally likely. Find the probability that the first person receives all four aces.

Solution:

The total number of choices is $\binom{52}{26}$. The number of possibilities where the first person receives all four aces is equal to the number of ways to distribute the remaining 48 cards, with 22 going to player 1 and 26 going to player 2, which is $\binom{48}{22}$. Thus, the answer is

$$\frac{\binom{48}{22}}{\binom{52}{26}}.$$

Problem 6: (10 points)
n male and n female dinner guests sit randomly on a long linear table with $2n$ seats. For any k in the range $1\ldots,2n-1$, we say that the pair of seats k and $k+1$ is "'interesting" if the two seats are occupied by guests of different genders. Let A_k be the event that the pair $(k,k+1)$ is interesting.

(a) Find the probability that the pair $(k,k+1)$ is interesting.

(b) Find the expected value of the number of interesting pairs.

(c) Are the events A_k and A_{k+2} independent?

Solution:

For part a, observe that there are $\binom{2n}{n}$ choices. How many of those choices have different genders on the k and $k+1$'s places? Given a male in the k'th spot and a female in the $k+1$'st, the number of ways to fill in the rest $\binom{2n-2}{n-1}$. Given a female in the k'th spot and a male in the $k+1$'st, the number of ways to fill in the rest is, again, $\binom{2n-2}{n-1}$. Therefore, the probability that the pair $(k,k+1)$ is interesting is

$$\frac{2\binom{2n-2}{n-1}}{\binom{2n}{n}} = \frac{n}{2n-1}.$$

For part b, let I_k be the indicator function of A_k. Then, the number of interesting pairs is $\sum_{k=1}^{2n-1} I_k$, and its expected value is

$$\sum_{k=1}^{2n-1} E[I_k] = (2n-1)\frac{n}{2n-1} = n.$$

For part c, consider the probability that both A_k and A_{k+2} to occur. There are four ways to put interesting pairs in both slots $(k,k+1)$ and $(k+2,k+3)$. The number of ways to fill in the remaining slots is $\binom{2n-4}{n-2}$, so that

$$P(A_k \bigcap A_{k+2}) = \frac{4\binom{2n-4}{n-2}}{\binom{2n}{n}},$$

which is not equal to $(\frac{n}{2n-1})^2$, so the events are not independent.

Problem 7: (10 points)
Let X and Y be positive continuous random variables with joint PDF $f_{X,Y}(x,y) = e^{-x-y}$, for $x>0$ and $y>0$.

(a) Let $V = X + Y$ and $U = XY$. Use the Jacobian formula to find the joint PDF of V and U.

(b) Are U and V independent?

Solution:

Solving for X, Y in terms of U, V we get the equation $U = X(V - X)$ which is a quadratic which leads to solutions

$$\begin{aligned}\max(X,Y) &= \frac{V + \sqrt{V^2 - 4U}}{2} \\ \min(X,Y) &= \frac{V - \sqrt{V^2 - 4U}}{2}\end{aligned}$$

Observe that its always true that $V^2 \geq 4U$ (since $(x+y)^2 \geq 4xy$ for all x, y), so that the above functions are indeed real-valued. This a one to one, continuous function mapping pairs (u, v) satisfying $v^2 - 4u \geq 0$ to pairs (M, m) satisfying $M \geq m$.

Thus, let us first change variables from (X, Y) to $M = \max(X, Y)$ and $m = \min(X, Y)$. We have

$$P(M \leq x, m \leq y) = 2P(X \leq x, Y \leq y) - P(X \leq y)^2 \text{ when } x \geq y,$$

and

$$P(M \leq x, m \leq y) = P(X \leq x)^2 \text{ when } x < y.$$

Differentiating with respect to both variables, we obtain:

$$f_{M,m}(x,y) = 2e^{-x-y}, \text{ when x} \geq \text{y} .$$

Now we use the change of variables formula. The Jacobian is

$$J_{g^{-1}}(u,v) = \det \begin{pmatrix} -\frac{1}{\sqrt{v^2-4u}} & \frac{1}{2} + \frac{v}{2\sqrt{v^2-4u}} \\ \frac{1}{\sqrt{v^2-4u}} & \frac{1}{2} - \frac{v}{2\sqrt{v^2-4u}} \end{pmatrix} = -\frac{1}{\sqrt{v^2 - 4u}}$$

so that $|J| = 1/\sqrt{v^2 - 4u}$. Thus the answer is

$$f_{U,V}(u,v) = 2\frac{e^{-v}}{\sqrt{v^2 - 4u}}, \text{ for all } u, v \geq 0 \text{ satisfying } v^2 - 4u \geq 0.$$

Since this distribution does not factor, U and V are not independent.

Remark: There is a considerably simpler derivation of the Jacobian. We will compute the Jacobian of the forward transformation $(x, y) \to (u, v)$ and take its inverse. The forward Jacobian is

$$det \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix} = y - x.$$

Now observe that

$$(x - y)^2 = (x + y)^2 - 4xy = v^2 - 4u,$$

which immediately gives us $|J_{g^{-1}}| = 1/\sqrt{v^2 - 4u}$.